Technische Universität Braunschweig

Master's Thesis

Split Packing: An Algorithm for Packing Circles with up to Critical Density

Sebastian Morr

2016-06-02

Institute of Operating Systems and Computer Networks

Supervisors:
Prof. Dr. Sándor Fekete
Dr. Christian Scheffer

Abstract

In the classic *circle packing problem*, one asks whether a given set of circles can be packed into the unit square. This problem is known to be NP-hard. In this thesis, we present a new sufficient condition using only the circles' combined area: It is possible to pack any circle instance with a combined area of up to $\approx$53.90% of the square's area. This area condition is tight, in the sense that for any larger percentage, there are instances which cannot be packed. We call the ratio between the largest area that can always be packed and the area of the container the *critical density*. Similar results have long been known for squares, but to the best of our knowledge, this thesis gives the first results of this type for circular objects.

Our proof is constructive: We describe a versatile, divide-and-conquer-based algorithm for packing circles and other objects into various container shapes with up to critical density. It employs an elegant subdivision scheme which recursively splits the circles into two groups and then packs these into subcontainers. We call the algorithm *Split Packing*. Beside realizing all packings below the critical density bound, there is a second perspective on this algorithm: It can be used as a constant-factor approximation algorithm when looking for the smallest container in which a given set of circles can be packed, due to its polynomial runtime.

In this thesis, we demonstrate that the Split Packing approach is applicable to a large number of different packing problems. Beside for square containers, we are able to apply the algorithm to pack circles with critical density into right and obtuse triangles, equilateral triangles, isosceles triangles whose longest side is their base, and rectangles with an aspect ratio of more than $\approx 1.57 : 1$. Additionally, we show that the same algorithm can be used to pack objects other than circles: It can pack octagons into squares and isosceles right triangles with critical density, and if allowed to rotate the objects, it can even be used to pack squares into these containers with critical density.

Aside from the obtained results, we believe that the ideas behind Split Packing are interesting and elegant on their own. We see many opportunities to apply these techniques in the context of other packing and covering problems.

A browser-based, interactive visualization of the Split Packing approach and other related material can be found at `https://morr.cc/split-packing/`.

Acknowledgements

I would like to thank the following people for their support in the creation of this thesis:

- Christian Scheffer, for always taking time for me, for many fruitful discussions and for sharing my enthusiasm.
- Christiane Schmidt, your lecture on Computational Geometry in 2014 started it all.
- Dominik Krupke, for sharing an office, ideas, and Mate.
- Enrico Jörns, for creating the beautiful LaTeX template.
- Heike Laschin, for being my best friend, and for softening my life.
- Jan-Marc Reinhardt, for all your encouragement. You were the first one to whom I showed the Split Packing algorithm.
- Marco Nikander, for spending late hours with me pondering over covering problems.
- The Pinky Crew: Jakob, Leslie, Marc, Moritz, and Torsten, for sharing breakfast, time, and friendship.
- Sándor Fekete, for sharing your know-how all along, and for suggesting this awesome topic to me.
- Wolfgang Morr, for your love and your unconditional support in more ways than I can count.
- The proof-readers: Christian Neukirchen, Dominik Krupke, Jakob Garbe, Jan-Marc Reinhardt, Jan Steiner, Marc Kastner, and Roland Hieber, for fixing all my embarrassing mistakes.
- All participants of the 31rd Bellairs Winter Workshop on Computational Geometry. I drew much inspiration from every one of you.

Colophon

This document was created using LaTeX 2_ε by Leslie Lamport and uses the tubsLaTeX class by Enrico Jörns. The figures were created using Ti*k*Z by Till Tantau and the text is set in the Nexus font family by Martin Majoor.

Contents

1 Introduction

Suppose you are given a set of of circles. Can you decide whether it is possible to pack these circles into the unit square without overlapping one another or the square? This is known as the *circle packing problem*, and it is known to be NP-hard [9], which means that there is (probably, assuming $\mathsf{P} \neq \mathsf{NP}$) no deterministic polynomial-time algorithm which can make that decision.

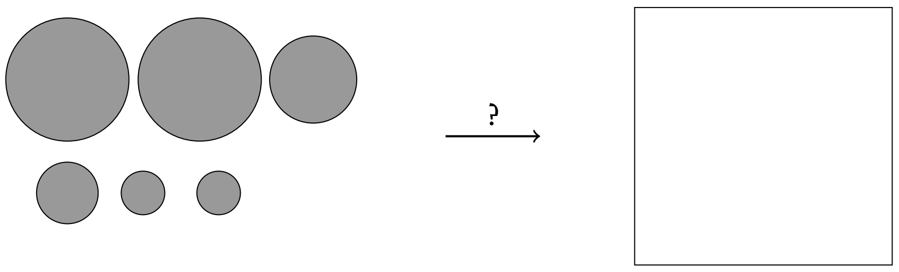


Figure 1.1: Can these circles be packed into the square?

Surely, there are necessary conditions for a packing to exist. For example, if the circles' combined area exceeds the area of the square, we can be certain that they can *not* be packed, as they will never fit inside the square without overlap. On the other hand, an upper bound on the circles' combined area could also serve as a *sufficient* condition. We may be able to say: "If the circles' combined area does not exceed some number a, then they can always be packed."

In fact, we could not only ask for *some* bound, but for the largest one. This is the problem which motivated this thesis:

Problem 1. What is the largest a so that any set of circles with a combined area of a can be packed into the unit square?

Already in 1967, Moon and Moser [34] solved this problem for square objects. They proved that it is possible to pack a set of squares into the unit square if their combined area does not exceed $1/2$. Their method first sorts the squares by size in descending order and then places them in the container square in a shelf-like manner, as depicted in Figure 1.2. At the same time, $1/2$ is the *largest upper bound* you could hope for. Consider the square packing in Figure 1.2 consisting of two equal squares, each of which has one fourth of the container square's area: If you made the squares only slightly larger, they could no longer be packed without overlapping one another.

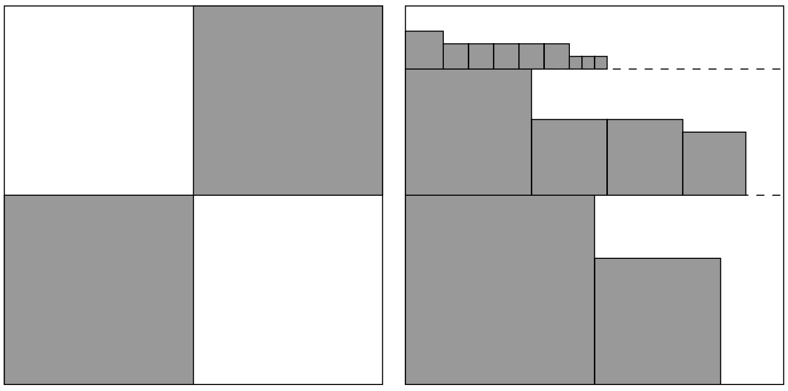


Figure 1.2: Left: Worst-case instance for packing squares into a square. Right: Example for Moon and Moser's shelf-packing.

The equivalent problem for packing circles has since been open. When Demaine, Fekete, and Lang [9] posed Problem 1 in 2010, they already suggested that the critical density may be limited by the two-circle instance shown in Figure 1.3. Again, it is easy to argue that if these two circles were only a little larger, we could no longer pack them into the unit square without overlap. This means that their combined area represents an upper bound on the area which can always be packed.

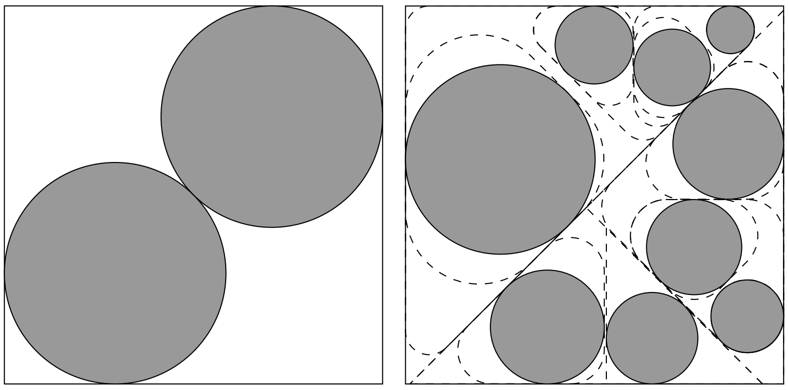


Figure 1.3: Left: Conjectured worst case for packing circles into a square. Right: Example packing produced by the Split Packing algorithm.

In this thesis, we will give a constructive proof that you can indeed pack all circle instances whose area does not exceed the combined area of the two circles shown in Figure 1.3 into the unit square. See Figure 1.3 for an example packing. The method is based on a divide-and-conquer approach which repeatedly splits the circle instance in halves, and then packs them recursively. In Figure 1.4, we demonstrate how the subdivision of the square into subcontainers looks like for an example instance.

We will also generalize this packing approach into two directions: On the one hand, Split Packing can be used to pack circles into other containers than squares, like certain classes of triangles and rectangles. On the other hand, we will demonstrate that it also can be used to pack other objects than circles.

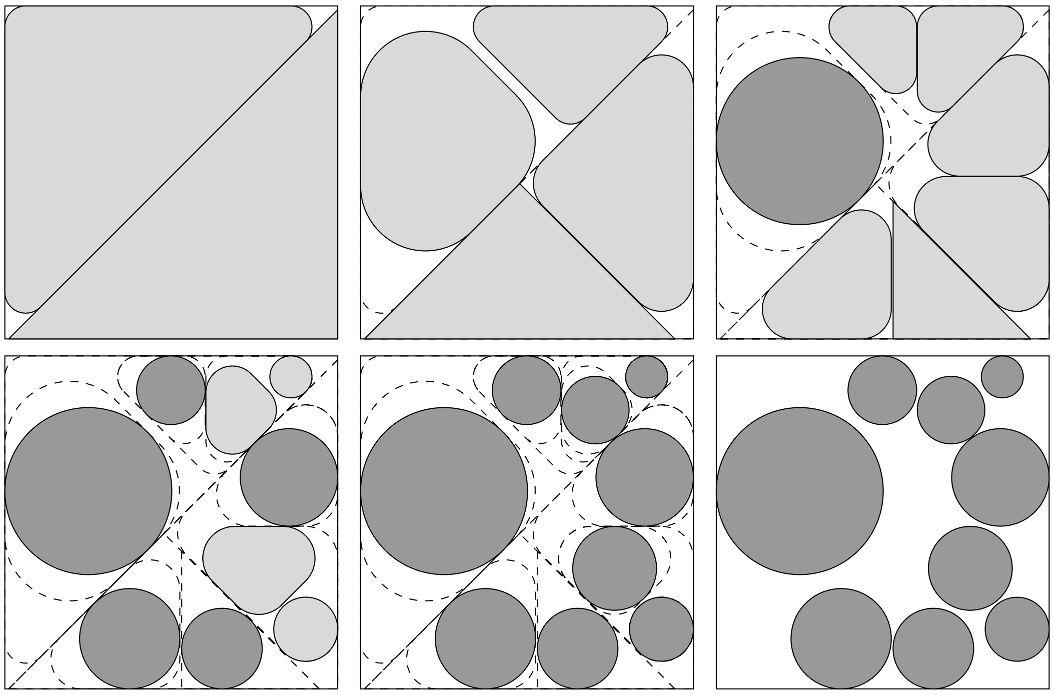


Figure 1.4: Split Packing recursively subdivides the container into subcontainers (light gray), before packing the circles into them (dark gray).

1.1 Results

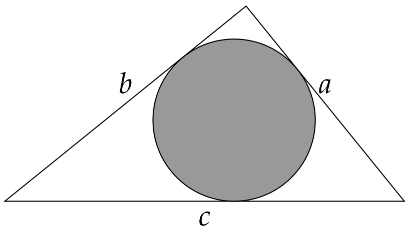
In Tables 1.1 and 1.3 we summarize our results. Each table entry represents a different packing problem, namely a specific combination of an object shape (like circles or squares), and a container shape (like a triangle or a rectangle).

For each problem, we call the ratio between the maximally packable area and the container's area the *critical density*. For all packing problems listed, we will give a constructive proof that any object instance can be packed into the container using rotations and reflections if the objects' combined area does not exceed this critical density. Each table entry contains a graphical representation of the worst-case instance which defines the critical density.

Split Packing can also be used as an approximation algorithm of the smallest-area container which can pack a given set of objects. The constant approximation factor ALG/OPT is equal to the reciprocal of the critical density. It is also listed in the table.

Note that we restrict some container types to fulfill certain conditions. For example, our results for packing circles into triangles only hold if the container is a right or obtuse triangle.

Table 1.1: Overview of results for packing circles.

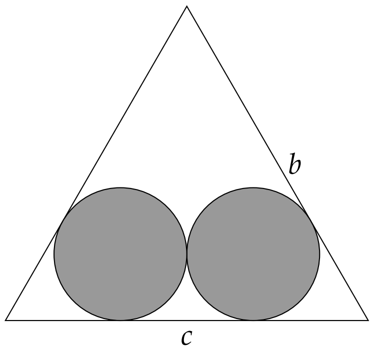


Circles in a right/obtuse triangle (Theorem 6)

Condition: $a^2 + b^2 \leq c^2$

Density: $\sqrt{\frac{-(a-b-c)(a+b-c)(a-b+c)}{(a+b+c)^3}}\pi < 53.91\%$

Approximation factor: Larger than 1.8552

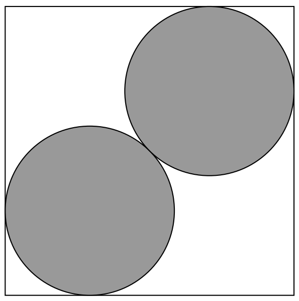


Circles in a thick isosceles triangle (Theorem 7)

Condition: $\frac{c}{\sqrt{2}} \leq b \leq c$

Density: $48.60\% < \frac{(c-2b+\sqrt{4b^2-c^2})^2\pi}{2c\sqrt{4b^2-c^2}} < 53.91\%$

Approximation factor: Between 1.8552 and 2.0576

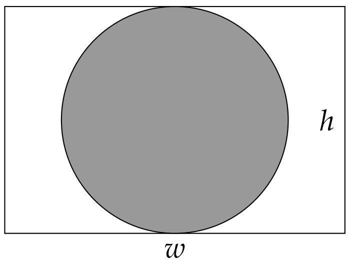


Circles in a square (Corollary 2)

Condition: –

Density: $\frac{\pi}{3+2\sqrt{2}} \approx 53.90\%$

Approximation factor: $\frac{3+2\sqrt{2}}{\pi} \approx 1.8552$



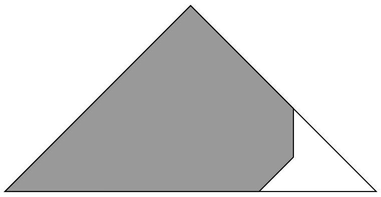
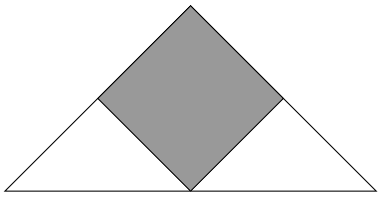
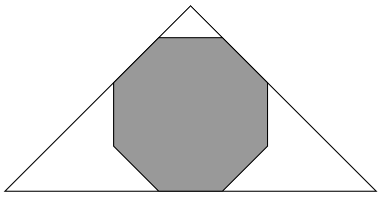
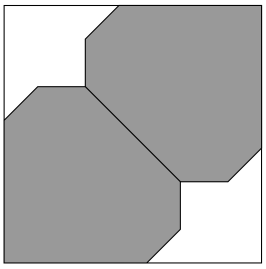
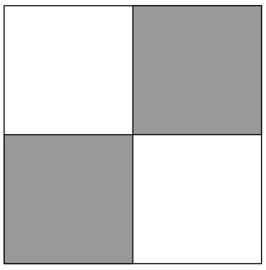
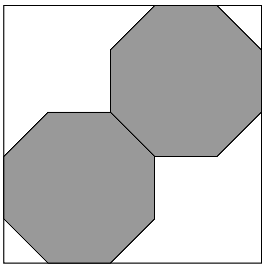
Circles in a long rectangle (Theorem 8)

Condition: $w \geq \frac{2+3\sqrt{2}}{4}h \approx 1.5607h$

Density: $\frac{\pi h}{4w} < 50.33\%$

Approximation factor: $\frac{4w}{\pi h} > 1.9870$

Table 1.3: Overview of results for packing other object types.

	**“Sharp rubies” in an isosceles right triangle** (Theorem 11) *Density:* $4\sqrt{2(\sqrt{2}-1)}+3\sqrt{2}-7 \approx 88.34\%$ *Approximation factor:* ≈ 1.1320
	**Squares in an isosceles right triangle** (Corollary 1) *Density:* 50% *Approximation factor:* 2
	**Octagons in an isosceles right triangle** (Corollary 1) *Density:* $8(5\sqrt{2}-7) \approx 56.85\%$ *Approximation factor:* ≈ 1.7589
	**“Rubies” in a square** (Theorem 12) *Density:* $8\sqrt{2(\sqrt{2}-1)}+6\sqrt{2}-15 \approx 76.67\%$ *Approximation factor:* ≈ 1.3043
	**Squares in a square** (Corollary 2) *Density:* 50% *Approximation factor:* 2
	**Octagons in a square** (Corollary 2) *Density:* $8(5\sqrt{2}-7) \approx 56.85\%$ *Approximation factor:* ≈ 1.7589

1.2 Applications

Circle packing is a natural, intuitive problem that has numerous applications in engineering, science, and everyday life.[1]

Packaging An obvious application is packaging: Given a set of cylindrical objects (like bottles or cans), one may want to determine the minimum-volume parcel in which all objects can be packed [7]. Or, given a shipping container of fixed size, one may want to pack as many rolls of paper into it as possible [13].

Bundling A similar application involves cylinder bundles: One may want to run several optical fibers through a tube, or place multiple pipes with different diameters inside of a larger one [43]. As another example, a car manufacturer may want to drill circular holes into a car's body and route several sensor cables through that hole, while keeping the hole as small as possible [40].

Cutting industry Circle packing is applicable in the cutting industry, where one may want to cut circular pieces from a material, minimizing waste [41]. A related problem is the layout of control panels containing circular controls (like in airplane cockpits, for example), which should be placed space-efficiently [7].

Communication In the context of digital modulation schemes, *quadrature amplitude modulation* is using circle packings to divide the two-dimensional phase-amplitude space into regions, which encode different binary patterns. Larger circles lead to increased noise tolerances [36]. Network planners may want to place radio towers in a geographical region, minimizing interference, but maximizing coverage [41].

Chemistry Other applications stem from chemistry, where one can study dense packings of atoms in crystals or macromolecules [44].

Foresting In foresting, one may want to plant trees so that the grown forest will be as dense as possible, but the trees do not hinder one another's growth [41].

Design A surprising application of circle packing involves the design of crease patterns for origami. In this problem, one wants to determine a sequence of folds of a square piece of paper, so that the result resembles a specific shape. The design of origami structures which resemble trees involves a packing problem called circle/river-packing. See Figure 6.2 on page 60 for an example. This kind of origami design was pioneered by Robert J. Lang [26]. Using circle packings only (without "rivers"), one can design crease patterns for folding arbitrary stars.

[1]Note that, in this thesis, we show a rather theoretical result of limited practical use. As Aleks commented on an example packing produced by Split Packing: "I'm sure this is how Amazon calculates packages. Buy something tiny, get it in a HUGE box." (`https://twitter.com/nd_cmptr/status/722139780608299009`)

1.3 Organization

This rest of this thesis is structured as follows: First, in **Chapter 2**, we give an overview of related work regarding the packing of squares and circles. In **Chapter 3**, we describe the general Split Packing algorithm and analyze its properties. We then proceed to show how Split Packing can be applied to several different packing problems: In **Chapter 4**, we discuss problems involving circular objects, and in **Chapter 5**, we show that the algorithm can even be generalized to pack objects other than circles. Finally, in **Chapter 6**, we show some directions for future work.

2 Related work

The literature on packing can roughly be divided into three categories, depending on the authors' objective:

Some authors strive to obtain the **optimal solutions** to packing problems. As almost all interesting packing problems are NP-hard, this becomes incredibly computationally expensive for more than a few objects. So, another approach is to design fast **heuristics** in order to obtain "good" solutions. These solutions are often close-to-optimal, but may degenerate for certain instances and thus make no guarantees about the actual quality of the produced packings. Finally, some authors develop **approximation algorithms**, which do make performance guarantees about the obtained solutions. In the best case, these algorithms produce solutions which are only a constant factor away from the optimal ones. This last approach is the one which we also follow in this thesis. There has been some work regarding approximation algorithms for packing rectangular objects, but similar problems with circular objects have not received much attention so far.

A packing problem consists of three parts: The first part is the type of objects that are to be packed. For example, these can be square or circular objects, and they can be allowed to have different sizes or be restricted to be of equal size. The second part is the type of the container, for example, a square, a triangle, or a general polygon. Thirdly, one can define restrictions on the movement of the objects: The objects could be allowed to rotate, or be restricted to translations only. Throughout this thesis, we allow the objects to be translated, rotated and reflected unless mentioned otherwise. There are also online variants of packing problems, where the object instance is not known beforehand, but for the purpose of this thesis, we restrict ourselves to offline approaches.

In general, packing arbitrarily-shaped objects into arbitrary containers can immediately shown to be NP-hard: Consider a rectangular container of dimensions $2 \times n$ and rectangular objects of height 1, whose widths add up to $2n$. The decision problem whether it is possible to pack these objects into the container yields a reduction from Partition.

2.1 Packing squares

We begin by briefly reviewing related work on packing squares into various containers. Algorithms which pack squares are interesting to us in the context of this thesis as we can directly use them for packing circles: We replace each circle of radius r with a square of edge length $2r$ and pack the resulting squares using one of the square-packing algorithms to obtain a packing of the original circle instance. In a sense, square-packing approximation algorithms provide us with lower bounds on what more specialized circle-packing approximation algorithms should be able to accomplish.

2.1.1 Hardness

We get a seemingly simple packing problem by restricting the container as well as the objects to be (not necessarily equal) squares. But even the decision problem whether it is possible to pack a given set of squares into a square container was shown to be strongly NP-complete by Leung et al. [27] through a reduction from 3-Partition.

For another variant, we could restrict the square objects to be equally sized, but allow the container to be arbitrary. In this case, too, the decision problem remains hard: Fowler, Paterson, and Tanimoto [12] showed it to be NP-hard to pack equal squares into arbitrary polygons with holes by a reduction from 3-Sat.

2.1.2 Approximation algorithms

As mentioned in the introduction of this section, we can directly use approximation algorithms for packing squares to obtain approximation algorithms for packing circles. If we are given an algorithm which can pack all square instances with a combined area of a into a given shape, we immediately get an algorithm which can pack all circle instances with a combined area of $\frac{\pi}{4}a \approx 0.7854a$ into that shape.

Moon and Moser [34] developed a packing algorithm for squares with a tight density bound: Their algorithm can pack all square instances with a combined area of $1/2$ into the unit square. It works by sorting the squares by their size and placing them into the square in a shelf-like manner, see Figure 1.2 on page 2. But they also proved that $1/2$ is the critical density when packing squares into a square: Two squares, each with an area of more than $1/4$ of the container's area, cannot be packed. As by our earlier remark, this algorithm can be used to pack circles: It can pack all circle instances with a combined area of $\frac{\pi}{8} \approx 0.3927$ into the unit square.

Kleitman and Krieger [25] showed that any square instance with a combined area of 1 can be packed into a rectangle of dimensions $\frac{2}{\sqrt{3}} \times \sqrt{2} \approx 1.1547 \times 1.4142$, and that this is the smallest-area rectangle with that property. The packing density in this case is about 0.6124. Applied to circles, this result means that all circle instances with a combined area of 1 can be packed into a rectangle of dimensions $\frac{4}{\sqrt{3\pi}} \times \frac{2\sqrt{2}}{\sqrt{\pi}} \approx 1.3030 \times 1.5958$, leading to a packing density of about 0.4810.

Hougardy [23] recently showed that for any set of squares with a combined area of 1, there is some rectangle with an area of less than 1.4 in which the squares can be packed using a computer generated proof. This means that for any set of circles with a combined area of 1, there is some rectangle with an area of less than $\frac{28\pi}{5} \approx 1.7825$ in which the circles can be packed.

2.2 Packing circles

There is a large body of work regarding the packing of circles into various containers. Most authors have considered the packing of equal circles into squares, see Szabó et al. [41] for an overview of numerous heuristics and optimization methods. Hifi and M'hallah [21] survey the literature on packing unequal circles into squares, rectangles, circles and other shapes, as well as on packing spheres into three-dimensional containers.

2.2.1 Hardness

The decision problem whether it is possible to pack a given set of unequal circles into a given square, a rectangle, or an equilateral triangle was shown to be NP-complete by Demaine, Fekete, and Lang [9], by a reduction from 3-Partition: They build a circle instance which first forces some symmetrical free "pockets" in the resulting circle packing. The instance's remaining circles can then be packed into these pockets if and only if the related 3-Partition instance has a solution.

2.2.2 Density bounds

The densest packing of equal circles in the plane can be easily guessed: Arrange them on a hexagonal grid, so that each circle touches six others. This arrangement results in a packing density of $\frac{\pi}{\sqrt{12}} \approx 0.9069$. An incomplete proof was given by Thue [42], which was later fixed by Fejes Tóth [10]. For a simple proof, see the paper of Chang and Wang [8].

For unequal circles, the optimal packing density can get arbitrarily close to 1: The circle instance can be chosen so that it can be packed in the manner of an Apollonian gasket (see Figure 2.1), or in another asymptotically space filling manner, like in the construction of Bourke [5].

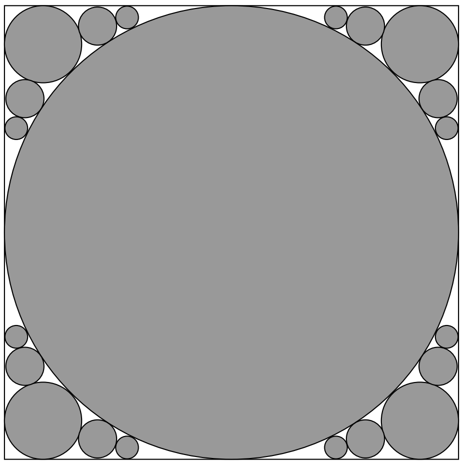


Figure 2.1: Circle instances which force a packing in the style of an Apollonian gasket approach a packing density of 1.

2.2.3 Optimal packings

A problem which has been considered a lot is the following: What is the smallest square in which n unit circles can be packed? Surprisingly, optimal results are only known for $n < 35$.

The optimal packings of up to nine equal circles into a square were already known in 1965: The cases for $n \leq 6$ are trivial, Schaer [37] proved the optimal solutions for $n = 7$ and $n = 8$, and Schaer and Meir [38] the one for $n = 9$.

The optimal solution for $n = 10$ was confirmed only in 1990, by Groot, Peikert, and Würtz [18]. In the same paper, they also prove the optimal solutions for 11, 12 and 13 circles. See Würtz, Monagan, and Peikert [44] for an overview of the history of optimal solutions for $n \leq 13$.

Some years later, Nurmela and Ostergård [35] provided computer-aided optimality proofs for $n \leq 27$. Their algorithm is based on a procedure by Peikert et al. [36]. Markót and Csendes [32] give computer-aided proofs for $n = 28, 29, 30$ within tight tolerance values. Finally, Locatelli and Raber [29] proved solutions for up to $n = 35$ to be optimal also within small tolerance values.

2.2.4 Approximation algorithms

When Demaine, Fekete, and Lang [9] posed the problem of finding the critical density for packing circles into a square (Problem 1), they also described a straightforward quadtree-based approximation algorithm which can guarantee a packing density of $\frac{\pi}{16} \approx 19.63\%$.

Miyazawa et al. [33] devised asymptotic polynomial-time approximation schemes for packing unequal circles into the smallest number of unit bins and for packing circles into a strip of unit width and of minimum height. They first pack "large" circles by solving a semialgebraic system, then cut the remaining free space into sub-bins, and pack the smaller circles into these bins by starting the same algorithm recursively.

Recently, Hokama, Miyazawa, and Schouery [22] developed an asymptotic approximation algorithm for the online version of the circle bin packing problem. Their algorithm also differentiates between large and small circles: They pack large circles according to the best known packings of equal circles, whereas small circles are placed in recursively subdivided hexagonal bins. This approach has an asymptotic competitive ratio of at most 2.4394. They also give a lower bound of 2.2920 for any bounded space algorithm for that problem.

2.2.5 Heuristics

For instances that contain too many circles to determine optimal packings, one can ask for lower bounds on the density of optimal packings; that is, the smallest *known* containers into which the circles can be packed. Many authors employ heuristics to produce good solutions for circle packing problems. A considerable amount of work has been put into finding new records, mostly for equal circles. The best known solutions for packing equal circles into squares, circles, rectangles, and other containers are continuously published on Specht's website `http://packomania.com` [39].

The works related to this idea are too numerous to be covered here in their entirety, but we will present some of the more important attempts. As described, these approaches do not give performance guarantees, but give lower bounds on the optimal packing of single instances. Most of the works mentioned here have lead to improvements of the best known packings in the literature at the time they were published.

Quasi-human and quasi-physical George, George, and Lamar [14] developed a set of heuristics for packing unequal circles into rectangles, based on enumerating *stable* solutions, where a circle is either on the ground or has two lower contacts. To get out of local optima, they evaluate a variety of approaches, like "shaking down" circles or applying genetic algorithms to improve the stable solutions.

Graham et al. [16] propose a *billiard simulation* approach, in which the circles are physically simulated as hard disks, and then apply some additional steps to tighten the found packings.

Boll et al. [4] model the disks as moving particles with repelling forces, and simulate them using a strategy resembling simulated annealing.

Wang et al. [43] describe a "quasi-physical, quasi-human" approach, which first simulates the circles according to gravity, then finds circles which are "most squeezed", and re-inserts them randomly.

Huang et al. [24] apply a quasi-human heuristic, which places new circles in gaps of approximately the same size as the circle, and combine it with a self-look-ahead strategy which evaluates how beneficial the choice of a certain placement for each circle is.

López and Beasley [30] apply a metaheuristic called *formulation space search*, which alternates between two different heuristics, to avoid getting stuck in local minima. After that, they finish off with an improvement phase, which swaps circles if this improves the solution.

He, Huang, and Yang [20] start with a random configuration of unequal circles in a square, simulate them using a quasi-Newton method to reach a minimum energy state, and then perform swapping steps, to get out of locally optimal solutions, followed by a postprocessing step to increase the result's precision.

Pattern-based Graham and Lubachevsky [17] identify various regular patterns for packing equal circles into a square, which produce close-to-optimal results for instances where the number of circles is equal to certain expressions like k^2, $k^2 - 1$, or $k(k+1)$.

The same authors [31] studied *curved hexagonal packings* of equal circles into a circle, a pattern derived from known best packings of circles in hexagons. For large n, a curved hexagonal packing was found to be suboptimal. Instead, good packings found by the authors seem to have a curved hexagonally-packed area in the center and an irregular pattern along the periphery.

2.2.6 Optimization methods

Another much-explored idea is to model the packing problem as a set of (mostly nonlinear) inequalities and to solve this system using commercial solvers.

Stochastic optimization Zhang and Deng [45] combine simulated annealing and tabu search to pack unequal circles into a circle.

Flores, Martínez, and Calderón [11] solve the problem of packing unequal circles into a circle using genetic algorithms and other evolutionary approaches. As the evolved solutions are not always valid, they evaluate different repair strategies, like repulsion-based or Delauny-based methods.

Addis, Locatelli, and Schoen [1] model the problem of packing circles into a square as a global nonlinear optimization problem and apply a stochastic search method similar to Multistart, which they refer to as *Monotonic Basin Hopping*, and which is inspired by molecular conformation problems. They also adapted this algorithm to pack circles into circular containers [19].

Linear optimization Litvinchev, Infante, and Ozuna Espinosa [28] compute approximated solutions to packing circles into rectangles by restricting their coordinates to a regular grid and formulate a binary Linear Problem, where the variables represent the assignment of the circles' centers to the grid's nodes.

Nonlinear optimization Birgin and Sobral [3] design packing problems as a smooth nonlinear programming model, and solve them using continuous optimization. They give multiple formulations of the problem, allowing them to pack different-sized circles into circles, squares, strips, rectangles or equilateral triangles and to pack spheres into spheres, cuboids, tetrahedra, pyramids and cylinders.

Birgin and Gentil [2] note that, given the contacts of the circle objects with the container's boundary beforehand, the number of constraints can be reduced significantly. They learn this contact information from the previously best known solutions in the literature and solve the now overdetermined linear system to get high-precision results.

3 Split Packing

While trying to prove the critical density for packing circles into a square, we worked on the assumption that it would in fact be possible to pack all circle instances whose area would not exceed that of the two-circle instance shown in Figure 1.3 on page 2. When we attempted to find a strategy which could pack this area, many strategies turned out unsuccessful. For example, a shelf-packing based approach inspired by Moon & Moser's algorithm for packing squares does not work for five equal circles, see Figure 3.1.

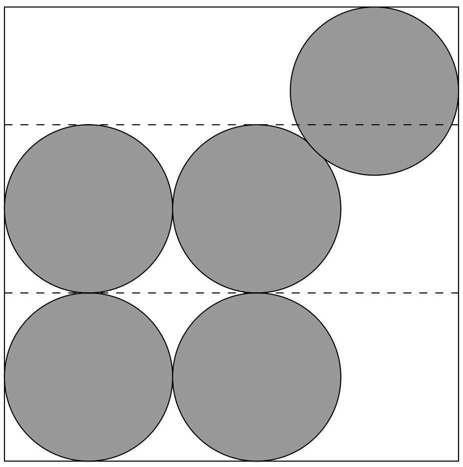


Figure 3.1: Shelf-packing does not work for five equal circles. The upper right circle slightly overlaps the middle circle.

At one point, we tried restricting the input instance with the hope that this would make the problem easier. We decided to only allow circles which have areas equal to a negative power of two of the total available area. This would only allow half-, quarter-, eight-, sixteenth-circles, and so on, in terms of the combined circle area a. We were surprised to find that these instances were easy to pack!

Imagine starting with the total area a combined into one large circle. It is easy to check that this circle can be packed, by placing it in the middle of the square, see the upper left of Figure 3.2. Now, we are allowed to "split" the circle into two half-circles, resulting in the worst-case instance discussed in the introduction. If we also cut the square into halves along the circles' tangent, both circles now are incircles of isosceles right triangles. If we now split any circle in half again, we can also divide its surrounding triangle into two smaller ones with half the area, and again the resulting circles are the triangles' incircles. This allows us to recurse, and to repeat the splitting as often as necessary, until we arrive at the desired circle instance. This divide-and-conquer approach to recursively split the set of input circles into subsets is the first key idea of the Split Packing algorithm.

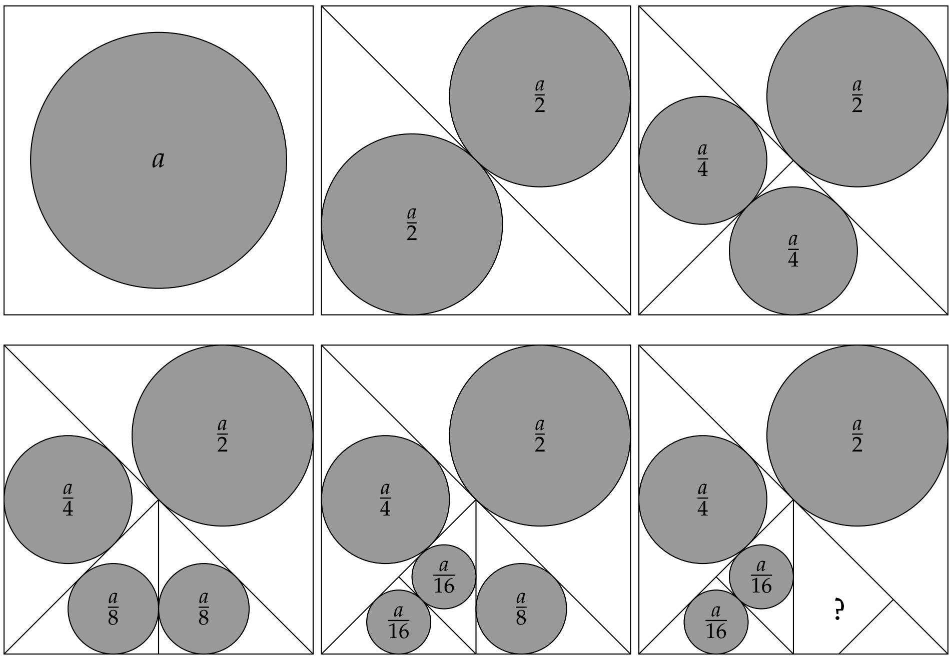


Figure 3.2: Splitting circles in half is easy.

If we now want to drop the restrictions on the circles, and allow arbitrary instances again, we run into a problem: What if we need to split a circle, but we do not want to split it in half? Instead, we may want to make one of the resulting circles larger than the other one. We would have to shift our cut to the side, like in the lower right of Figure 3.2. This subdivides the triangle into another (smaller-than-half) triangle and a quadrilateral. For the small triangle, we can start another recursion, but it is unclear why we can continue with the recursion for the quadrilateral. Its shape resembles our triangles, but the lower corner is "cut off". We would need an argument why these "degenerate triangles" do not break our packing strategy.

This is where the second key idea of the Split Packing algorithm comes into play: When performing the top-level split, we could already decide which circles of the input instance should go on which side of the cut-line. With the power-of-two instances, it did not matter at all *how* we split the circle set in half, as we could always split it into equally-sized halves. But for general circle instances, we need to proceed more carefully. We are going to perform the splitting using an algorithm which resembles greedy scheduling. This will make sure our resulting subgroups are *close* to equal in terms of their combined area. But if the groups' areas deviate from the targeted 1:1 ratio, we can make additional statements about the minimum circle size in the larger group, leading to a rounded subcontainer which will make the "cut-off corners" irrelevant.

In the rest of this chapter, we will detail this process. As we will see, the algorithm also generalizes to other container types and object shapes.

3.1 Definitions

First, we introduce a few basic notions, mostly to simplify some of the later proofs.

Definition 1. For the purpose of this thesis, a *shape* is either an open circle or an open polygon ("open" meaning that the shape's boundary is not part of the shape).

What it means to pack objects into a container is defined as one would intuitively expect:

Definition 2. A *packing* of a set of shapes (the *objects*) into another shape (the *container*) is an assignment of a translation, a rotation and a reflection to each shape so that

- the pairwise intersections of all objects are empty and
- all objects are a subset of the container.

We continue with some definitions which make it easier to talk about classes of circle instances:

Definition 3. For each $0 \leq a$, an *a-circle* is a circle with an area of a.

Definition 4. A *circle instance* is a multiset of nonnegative real numbers, which define the circles' areas. For any circle instance C, $\operatorname{sum}(C)$ is the combined area of the instance's circles and $\min(C)$ is the area of the smallest circle contained in the instance.

For example, for the circle instance $C = \{1, 1, 2, 3\}$, $\operatorname{sum}(C) = 7$ and $\min(C) = 1$.

Definition 5. $\mathbb{C}$ is the set of all circle instances. $\mathbb{C}(a)$ consists of exactly those circle instances C with $\operatorname{sum}(C) \leq a$. Finally, $\mathbb{C}(a, b)$ consists of exactly those circle instances $C \in \mathbb{C}(a)$ with $\min(C) \geq b$.

Let us give another example for the previous definition, as it is crucial for the rest of this thesis: For any circle instance C contained in $\mathbb{C}(1, \frac{1}{8})$, the combined area of C's circles is at most 1, and at the same time, each of C's circles is at least an eight-circle.

Finally, we introduce two definitions which come in handy when discussing worst-case instances:

Definition 6. A shape's *incircle* is the largest circle that can be packed into the shape.

Definition 7. A shape's *twincircles* are the largest two equal circles that can be packed into the shape.

3.2 Greedy splitting

The method which we use to split the circle instances in half, Split (Algorithm 1), resembles a greedy scheduling algorithm, which is why we call the process *greedy splitting*. The algorithm takes two inputs: The first is an arbitrary circle instance C, which is to be split into two groups. The circles are assumed to be sorted by size in descending order.

The second input is a tuple of two positive factors, which determines the targeted ratio of the resulting groups' combined areas. For example, if we wanted to split C into equally sized halves, we could choose the tuple $(1, 1)$. The tuple $(\frac{1}{2}, \frac{1}{2})$ would give the same result. For asymmetric containers, we may want to target a different ratio. For example, if we wanted to make one group three times as large as the other, we could use the tuple $(1, 3)$. We call these tuples *split keys*:

Definition 8. A *split key* is a tuple of two positive real numbers.

The algorithm first creates two empty "buckets", and in each step adds the largest remaining circle of the input instance to the "more empty" bucket. We say that a bucket is relatively more empty if the ratio between the sum of its current content and the associated factor of the split key is smaller than the ratio in the other one. In the simplest case, the split key will actually describe the desired areas of the two groups, and Split puts the next circle into the group which has the smaller "relative filling level".

Algorithm 1 Split(C, F)

```
Input: A circle instance $C$, sorted by size in descending order, and a split key $F = (f_1, f_2)$
Output: Circle instances $C_1, C_2$
  $C_1 \leftarrow \emptyset$
  $C_2 \leftarrow \emptyset$
  for all $c \in C$ do
      $j = \arg\min_i \frac{\mathrm{sum}(C_i)}{f_i}$          ▷ Find the index of the more empty bucket.
      $C_j \leftarrow C_j \cup \{c\}$
  end for
```

Although the greedy splitting aims to produce subinstances with area ratios close to the specification of the split key, of course it will not always be possible to achieve these ratios exactly. For example, if the input instance consists of few very large circles, it may even be impossible to distribute them in the specified ratios.

On the other hand, if the resulting groups' area ratio does deviate from the split key, we gain additional information about the "relatively larger" group: The more this group exceeds its targeted area, the larger the minimum size of its elements.

Lemma 1. For any C_1 and C_2 produced by Split$(C, (f_1, f_2))$:

$$\min(C_i) \geq \mathrm{sum}(C_i) - f_i \frac{\mathrm{sum}(C_j)}{f_j}$$

Proof. If $\frac{\mathrm{sum}(C_i)}{f_i} < \frac{\mathrm{sum}(C_j)}{f_j}$, then the lemma says that $\min(C_i)$ is larger than a negative number, which is certainly true.

Otherwise, set $r := \frac{\mathrm{sum}(C_j)}{f_j}$. This value describes the smaller "relative filling level" by the time the algorithm ends. Now assume for contradiction C_i contained an element smaller than $\mathrm{sum}(C_i) - f_i r$. As the elements were inserted by descending size, all elements which were put into C_i after that element would have to be at least as small. So the final element put into C_i (let us call it c) would be smaller than $\mathrm{sum}(C_i) - f_i r$, as well.

But this means that

$$\frac{\mathrm{sum}(C_i) - c}{f_i} > \frac{\mathrm{sum}(C_i) - (\mathrm{sum}(C_i) - f_i r)}{f_i} = r,$$

meaning that at the moment before c was inserted, the relative filling level of C_i would already have been larger than r. Recall that r is the smallest filling level of any group by the time the algorithm ends, meaning that at the time when c is inserted, C_i's filling level is already larger than the filling level of the other group. This is a contradiction, as the greedy algorithm would choose to put c not into C_i, but into the other group with the smaller filling level in this case. □

We can go even further: If one group's relative filling level becomes twice as large as the other one's, the group must consist of a single circle:

Lemma 2. For any C_1 and C_2 produced by $\textsc{Split}(C, (f_1, f_2))$:

$$\frac{\operatorname{sum}(C_i)}{f_i} > 2\frac{\operatorname{sum}(C_j)}{f_j} \Rightarrow \min(C_i) = \operatorname{sum}(C_i)$$

Proof. By Lemma 1,

$$\min(C_i) \geq \operatorname{sum}(C_i) - f_i\frac{\operatorname{sum}(C_j)}{f_j} > \operatorname{sum}(C_i) - f_i\frac{\operatorname{sum}(C_i)}{2f_i} = \frac{1}{2}\operatorname{sum}(C_i),$$

which means that the minimum size of the circles in C_i is already more than half of its total sum. The only possible instance which satisfies this constraint is a single circle with area $\operatorname{sum}(C_i)$. □

Note that we usually do need this last property and will ignore the additional constraint on $\min(C_i)$, except in Chapter 5, where we discuss packing objects other than circles.

We are now going to define a term which encapsulates all mentioned properties of the circle instances output by Split. These properties depend on the used split key, and also on the combined area a and the minimum circle size b of the circle instance, which is why it has three parameters. We say that instances produced by Split are (a, b, F)-*conjugated*:

Definition 9. For any $0 \leq b \leq a$ and any split key $F = (f_1, f_2)$, we say that the tuples (a_1, b_1), (a_2, b_2) are (a, b, F)-*conjugated* if

- $a_1 + a_2 = a$,
- $b_i \geq b$,
- $b_i \geq a_i - f_i\frac{a_j}{f_j}$, and
- $\frac{a_i}{f_i} > 2\frac{a_j}{f_j} \Rightarrow b_i \geq a_i$.

Definition 10. Two circle instances C_1 and C_2 are (a, b, F)-conjugated if there are any (a, b, F)-conjugated tuples (a_1, b_1) and (a_2, b_2) so that $C_1 \in \mathbb{C}(a_1, b_1)$ and $C_2 \in \mathbb{C}(a_2, b_2)$.

We can now associate this property with Split in the following theorem:

Theorem 1. *For any $C \in \mathbb{C}(a, b)$ and any split key $F = (f_1, f_2)$, $\textsc{Split}(C, F)$ always produces two (a, b, F)-conjugated subinstances.*

Proof. That the subinstances' combined areas add up to a follows directly from the algorithm. As the minimum size of all circles in C is b, this must also be true for the subinstances, so $\min(C_i) \geq b$. The other two minimum-size properties follow from Lemma 1 and Lemma 2. □

As described, one way to think about conjugatedness is that it gives guarantees for the minimum sizes of the "larger" produced subinstance. To give you an intuition of what the conjugatedness property will be used for in the later chapters, we show several examples of shapes with (a, b, F)-conjugated parameters in Figure 3.3: a_1 and a_2 represent the area which can be packed into the respective shape, while b_1 and b_2 represent their "rounding". The shapes can always be packed because if one shape gets larger, it is rounded so much that it still fits inside the container.

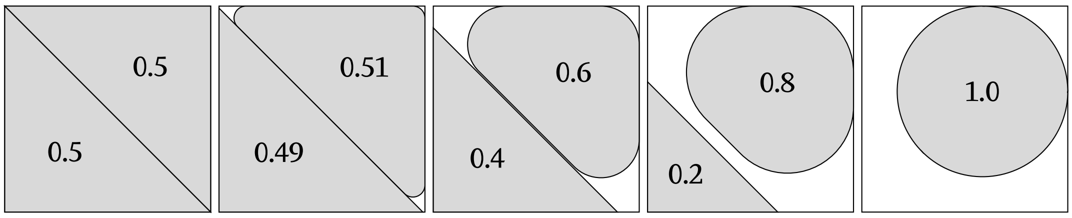


Figure 3.3: The two shapes' parameters are (a, b, F)-conjugated, which is why they always can be packed. The numbers represent a_1 and a_2, and in this case are the areas of the shapes' incircles.

3.3 Split Packing

The Split algorithm presented in the previous section, in addition to the properties of the instances it produces, are the foundations on which we now build the central theorem of this thesis: The Split Packing theorem. Split Packing by itself is a general approach to pack circles and other shapes into containers. The later chapters of this thesis will all apply the Split Packing theorem to different object and container types.

First, another definition. We will often want to state that a shape can pack all circle instances which belong to a certain class. For this, we define the term $\mathcal{C}$*-shape*:

Definition 11. For any $\mathcal{C} \subseteq \mathbb{C}$, a $\mathcal{C}$*-shape* is a shape in which each $C \in \mathcal{C}$ can be packed.

For example, if a shape is a $\mathbb{C}(a)$-shape, it means that it can pack all circle instances with a combined area of a. And a $\mathbb{C}(a, b)$-shape can pack all circle instances with a combined area of a, whose circles each have an area of at most b. With these preparations, we can now state our central theorem:

Theorem 2 (Split Packing)**.** *A shape s is a $\mathbb{C}(a, b)$-shape if there is a split key F, so that for all (a, b, F)-conjugated tuples (a_1, b_1) and (a_2, b_2) one can find a $\mathbb{C}(a_1, b_1)$-shape and a $\mathbb{C}(a_2, b_2)$-shape which can be packed into s.*

Proof. Consider an arbitrary $C \in \mathbb{C}(a, b)$. We use Split(C, F) to produce two subinstances C_1 and C_2. We know from Theorem 1 that those subinstances will always be (a, b, F)-conjugated. So if we can indeed find two shapes which can pack these subinstances, and if we can pack these two shapes into s, then we also can pack the original circle instance C into s.

Note that in the extreme case that C consists of a single circle, Split(C, F) will yield two circle instances $C_1 = \{C\}$ and $C_2 = \emptyset$. For this case, Theorem 1 guarantees a minimum size of a for the first group, and the associated $\mathbb{C}(a_1, b_1)$-shape is just an a-circle. This means that we can simply place the input circle in the container, and stop the recursion at this point. □

Written as an algorithm, Split Packing looks like this:

Algorithm 2 Splitpack(s, C)

```
Input: A $\mathbb{C}(a,b)$-shape $s$ and a circle instance $C \in \mathbb{C}(a,b)$, sorted by size in descending order
Output: A packing of $C$ into $s$
  Determine split key $F$ for shape $s$
  $(C_1, C_2) \leftarrow$ Split$(C, F)$                          ▷ See Algorithm 1.
  for all $i \in \{1,2\}$ do
      $a_i \leftarrow \mathrm{sum}(C_i)$
      $b_i \leftarrow$ minimum guarantee for $C_i$              ▷ See Definition 9.
      Determine a $\mathbb{C}(a_i, b_i)$-shape $s_i$
      Splitpack$(s_i, C_i)$
  end for
  Pack $s_1$, $s_2$, and their contents into $s$
```

To summarize, to apply Splitpack to a shape, we need the following elements:

1. a strategy to determine the shape-specific split key F,
2. a strategy to determine a $\mathbb{C}(a_1, b_1)$-shape and a $\mathbb{C}(a_2, b_2)$ for all (a, b, F)-conjugated tuples (a_1, b_1) and (a_2, b_2), and
3. a strategy to pack these two shapes into s, and a proof why this always works.

Note that the Split Packing algorithm can easily be extended to allow splitting into more than two subgroups. For simplicity, we only describe the case of two subgroups here, as this suffices for the shapes we discuss in this thesis.

3.4 Analysis

There are two perspectives on the implications of the Split Packing theorem. Suppose that we have all three elements mentioned above in place so that we can use Split Packing to pack each circle instance in $\mathbb{C}(a)$ into a given shape. First, we now have a sufficient condition for the decision problem if a given circle instance can be packed into a given container: If the circles have a combined area of at most a, then the instance can be packed.

Second, Split Packing can also be used as an approximation algorithm. Say you are given a circle instance of combined area a, but now we want to find the smallest container of a certain shape (for example, triangular or square) in which the instance can be packed. We can then use Split Packing as an approximation algorithm: Select a $\mathbb{C}(a)$-shape whose guaranteed packing density d is as high as possible. We can then be sure that this container has at most $\frac{1}{a}$ times the area of the optimal container.

We will first show that Split Packing has polynomial runtime, as required for an approximation algorithm, and then argue about the approximation factor.

Theorem 3. *Split Packing requires $\mathcal{O}(n)$ basic geometric constructions and $\mathcal{O}(n^2)$ numerical operations.*

Proof. To argue why we only need a linear number of geometric constructions, we can count the nodes of the recursion tree: Each subcontainer either has two children (if more than one circle needs to be packed inside, in which case a Split is performed), or one (in this case, the child is a single circle and the recursion ends). Without the circles, the recursion tree is a full binary tree with n leaf nodes, meaning that it has exactly $n-1$ interior nodes. The root node is the container of the packing problem, which does not need to be packed. In total, we need to pack $2n-2$ subcontainers, in addition to the n circles of the input instance, leading to $\mathcal{O}(n)$ geometric constructions.

In addition, to build the recursion tree, we need at most a quadratic number of numeric operations: Before SplitPack is first invoked, the circle instance has to be sorted by size in descending order, this can be done in $\mathcal{O}(n \log n)$ time. Additionally, each run of the Split subroutine then takes linear time in the size of its input. If Split would partition its input into two subinstances with a similar number of elements in each case, this would also lead to a runtime of $\mathcal{O}(n \log n)$, but in the worst case, each run only splits off one element, so that the total time needed for all Split operations is

$$n + (n-1) + (n-2) + \cdots + 1 \in \mathcal{O}(n^2).$$

□

Theorem 4. *Split Packing, when used to pack circles into a $\mathbb{C}(a,b)$-shape of area A, is an approximation algorithm with an approximation factor of $\frac{A}{a}$, compared to the container of minimum area.*

Proof. We know from the previous theorem that Split Packing has polynomial runtime.

As for the approximation factor, we can be sure that the area of the optimal container OPT needs to be as least a, as we need to be able to fit the circles inside without overlap:

$$\text{OPT} \geq a$$

At the same time, the area of the approximated container ALG is exactly A, which means that

$$\frac{\text{ALG}}{\text{OPT}} \leq \frac{A}{a}.$$

□

4 Packing circles

In the previous chapter, we gave a very general description of how Split Packing works. In this chapter, we will show how to apply the approach to concrete packing problems, namely, packing circles into certain triangles and rectangles. For each shape, we will demonstrate two properties: First, we will argue about the *worst-case circle instance* in terms of the density of its optimal packing. This instance constitutes an upper bound to the question "How much area can always be packed?", because we can show that, given more area, we can find instances which *cannot* be packed.

But second, we are going to show how Split Packing can be used to pack all instances of this area into the shape. This is a constructive way to show a lower bound to our problem. Both pieces together give the answer to Problem 1: We have found the largest area a, so that circles with this combined area can always be packed into the shape, and we have a polynominal-time algorithm that actually realizes the packing.

4.1 Hats

We saw in Chapter 3 that when applying SPLIT to a circle instance, we can make statements about the minimum circle size of the "relatively larger" group. If you think about how a circle of minimum size can be placed in a container, there may be certain points which can never be covered by a circle without overlapping the container's boundary. For example, if the container's boundary contains sharp angles, the minimum-size circle may be too large to cover all the angle's area. As we cannot use this for packing the circle instance at all, we may as well cut this area from the container without changing its packing capabilities.

Technically, you can calculate the usable area of a container shape by first eroding it by a circle of minimum size, and then dilating the result with the same circle.

This observation motivates a family of shapes which resemble rounded triangles. We call these shapes *hats*:

Definition 12. For each $0 \leq b \leq a$, an (a, b)-*hat* is a non-acute triangle with an incircle of area a, whose corners are rounded to the radius of a b-circle, see Figure 4.1. Call the two lower angles of the original triangle *left-angle* and *right-angle*. If we say *right hat* or *obtuse hat*, the hat is based on a right/obtuse triangle.

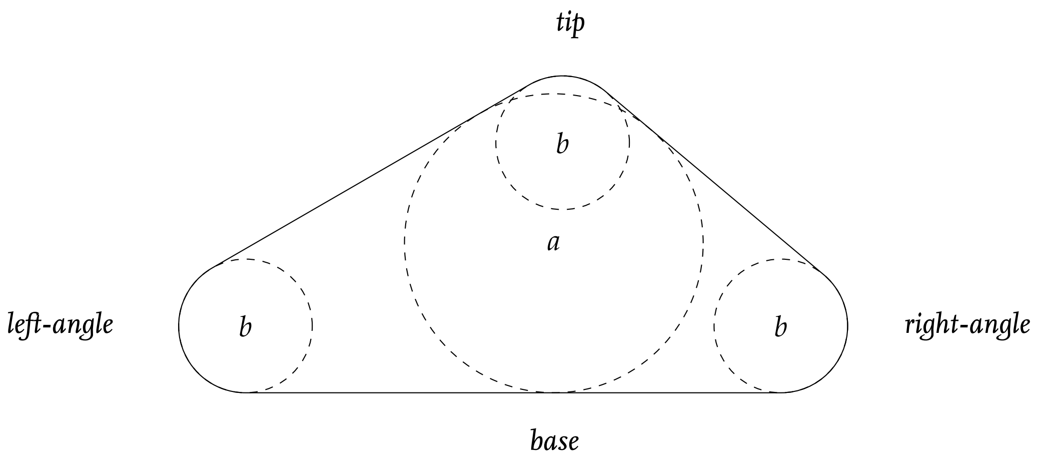


Figure 4.1: An (a, b)-hat.

In the rest of this section, we will show that all circle instances with a combined area of up to a with a minimum circle size of b can be packed into an (a, b)-hat.

Definition 13. To get a hat's *associated split key*, split the underlying triangle orthogonally to its base through its tip, and inscribe two circles in the two sides, see Figure 4.2. The areas of these circles are the two components of the hat's split key.

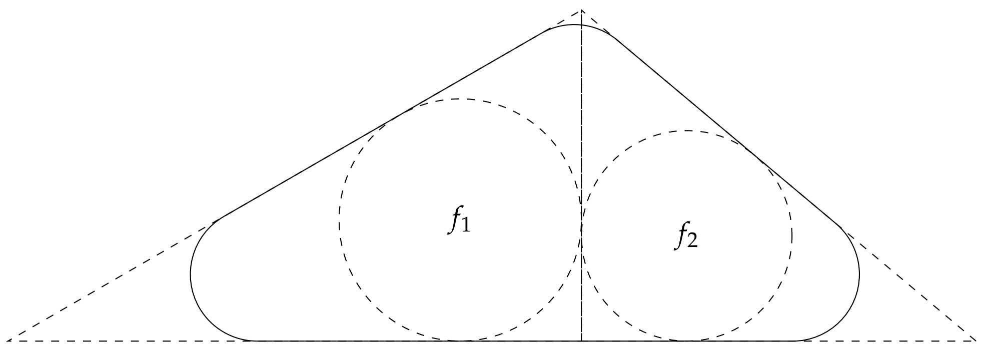


Figure 4.2: A hat's *associated split key* equals (f_1, f_2)

To prepare for the next proof, we establish the following lemma:

Lemma 3. Place two circles of combined area a in two corners of a triangle, like in Figure 4.3. Let w be the length of the connecting side of the triangle. Now, define p_1 and p_2 to be the "projection factors", so that, when projecting circle a_i down onto the connecting side, the distance between the triangle's corner and the far point of the projection is $\sqrt{a_i}p_i$. Note that these factors will be constant for arbitrary circle areas. The two projections do not intersect if

$$w \geq \sqrt{a(p_1^2 + p_2^2)}.$$

Proof. Let $w'(a_1) = \sqrt{a_1}p_1 + \sqrt{a - a_1}p_2$ be the combined width of both projections. This function has its global maximum at $a_1 = \frac{p_1^2}{p_1+p_2}a$, and the maximum value is $\sqrt{a(p_1^2 + p_2^2)}$. If w is at least as large as this value, the two projections do never intersect. □

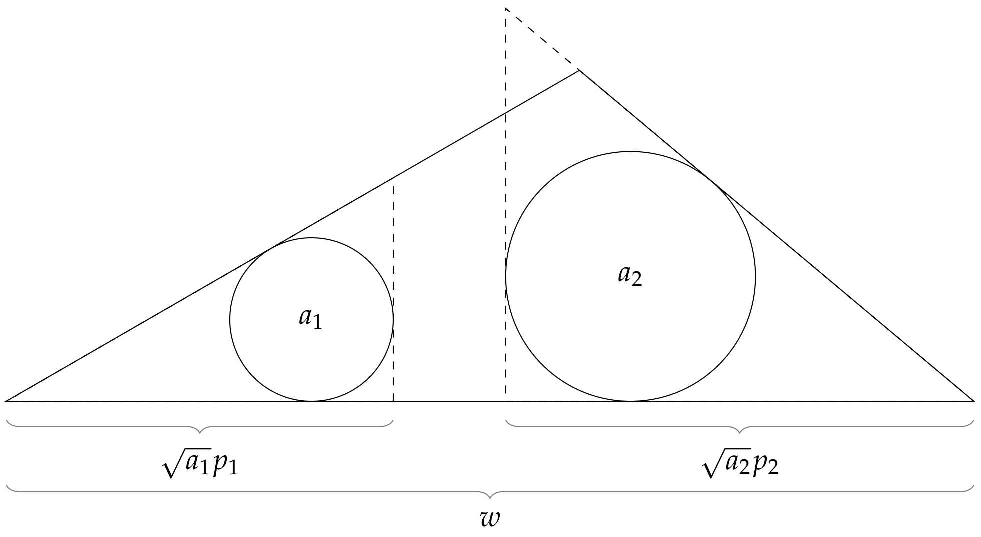


Figure 4.3: The circles' projections do not overlap if $w \geq \sqrt{a(p_1^2 + p_2^2)}$.

Lemma 4. Consider an $(a, 0)$-hat with the associated split key $F = (f_1, f_2)$, and call its left- and right-angles α and β. For all $(a, 0, F)$-conjugated tuples (a_1, b_1) and (a_2, b_2), the following two shapes can be packed into the hat:

- a right (a_1, b_1)-hat with a right-angle of α and
- a right (a_2, b_2)-hat with a left-angle of β.

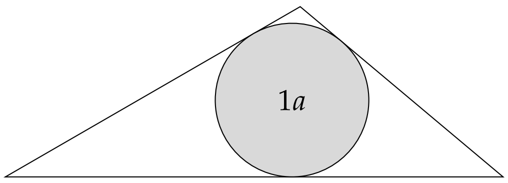
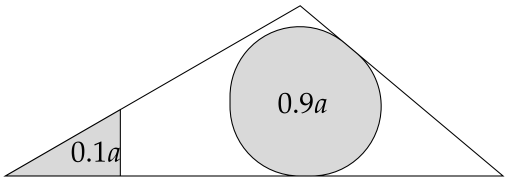
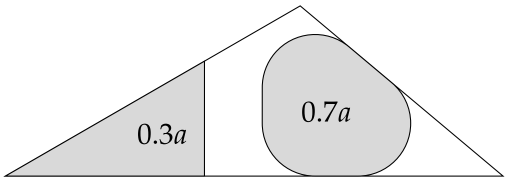
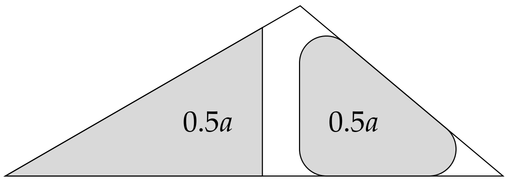
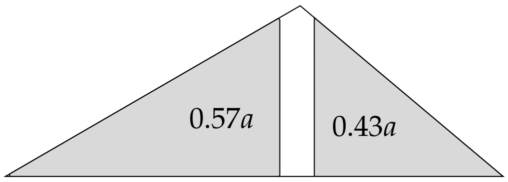
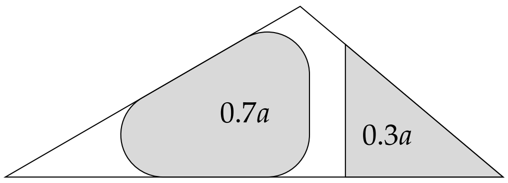
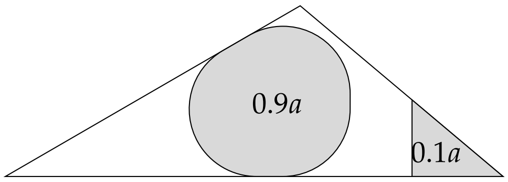
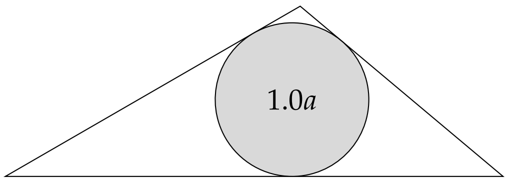
Proof. Place the hats' tips at the bottom of the container hat, rotate their α- and β-angles toward the container's matching angles and push them as far to the left/right as possible. Table 4.1 illustrates how these packings looks like for different values of a_1 and a_2 and gives some informal reasoning why they can always be packed.

This way of placing the two hats results in a valid packing if (1) the hats do not overlap each other and (2) the hats fit into the hat individually. We are going to proove these two properties separately.

We first want to show that the hats do not overlap each other. If the hats' projections onto the container's base do not overlap, we found a separating axis and can be sure that the hats do not overlap, as well. Furthermore, because the hats' incircles touch the rightmost part of the left hat's boundary and the leftmost part of the right hat's boundary, it suffices to show that the projections of the hats' incircles onto the container's base do not overlap.

We want to use Lemma 3 for this proof, so we need to make a statement about the projection factors p_1 and p_2 in Figure 4.4: If the top angle is a right angle, we can see that $\sqrt{a}p_1 = x$ and $\sqrt{a}p_2 = y$. So by the Pythagorean theorem, $w^2 = (\sqrt{a}p_1)^2 + (\sqrt{a}p_2)^2$. If the top angle is more obtuse, but the incircle's center stays at the same x-coordinate (like the dotted variant in Figure 4.4),

Table 4.1: Hat-in-hat packings for different values of a_1 and a_2.

	When $a_2 = a$, the second hat is simply rounded to a circle of area a. By definition, this is the incircle of the container, so it can be packed.
	As a_2 gets smaller, the second hat gets less rounded, but still enough so that it can be packed. The other hat is a non-rounded triangle, which fits beside the second one.
	
	
	When a_1 and a_2 have a ratio as specified by the container's associated split key, the two hats are both non-rounded triangles, but they can still be packed.
	When a_2 gets smaller than specified by the split key, it is now the first container which is getting more rounded.
	As a_1 approaches a, it is now the first hat whose shape gets closer to a perfect a-circle.
	By the same argument as before, this circle can be packed.

both $\sqrt{a}p_1$ and $\sqrt{a}p_2$ only get smaller, so for each hat, $w^2 \geq (\sqrt{a}p_1)^2 + (\sqrt{a}p_2)^2$, which is equivalent to $w \geq \sqrt{a(p_1^2 + p_2^2)}$. By Lemma 3, this means that the projections of the circles do not overlap, which in turn means that the two hats do not overlap.

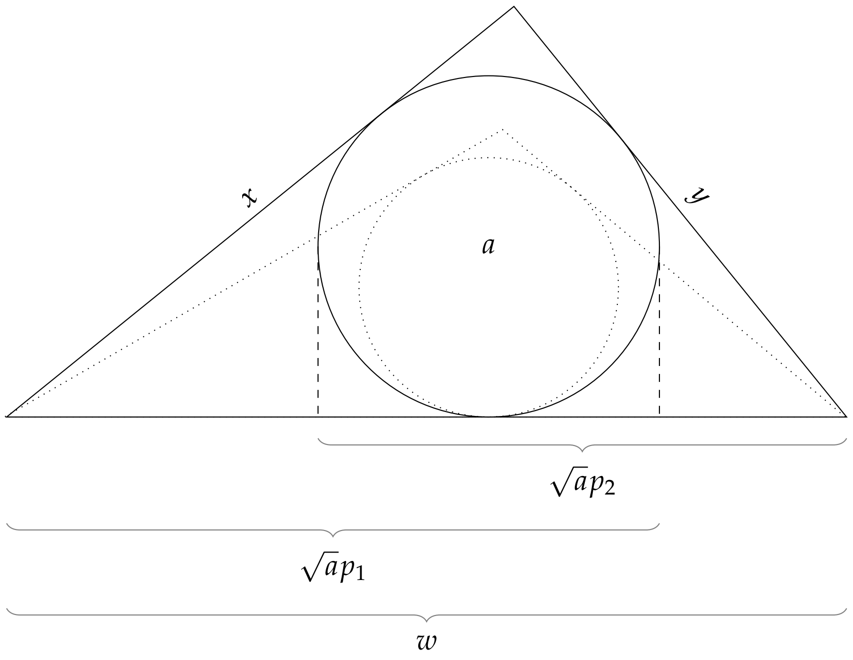


Figure 4.4: $l^2 + r^2 \leq 1$ holds for each non-acute triangle

The second property we need to show is that the hats fit into the container individually. Unfortunately, this part of the proof is going to be long and technical.

If a hat's incircle is not larger than the incircle of the container hat's side, it will fit into the container without question because it is a subset of that side (like all the non-rounded hats in Table 4.1). So let us assume $a_i > f_i$.

In this proof, we are going to use two different length-area ratios, which are illustrated in Figure 4.5. The first one is d, which describes the ratio between the length of the triangle's right leg and the square root of the area of its right incircle f_i. Note that for all triangles *similar* to the right part of the container triangle, this ratio between the length of this edge and the square root of the incircle's area will be constant. The second ratio, e, is the ratio between the length of the same right leg and the square root of the incircle-area of the whole container triangle. Again, it is a constant for triangles similar to the given container triangle. Note that, in preparation for a generalization later in this chapter, we denote the triangle's incircle by o. From Figure 4.5 we can now observe that $e\sqrt{o} = d\sqrt{f_i}$, which is equivalent to $e = d\sqrt{f_i/o}$.

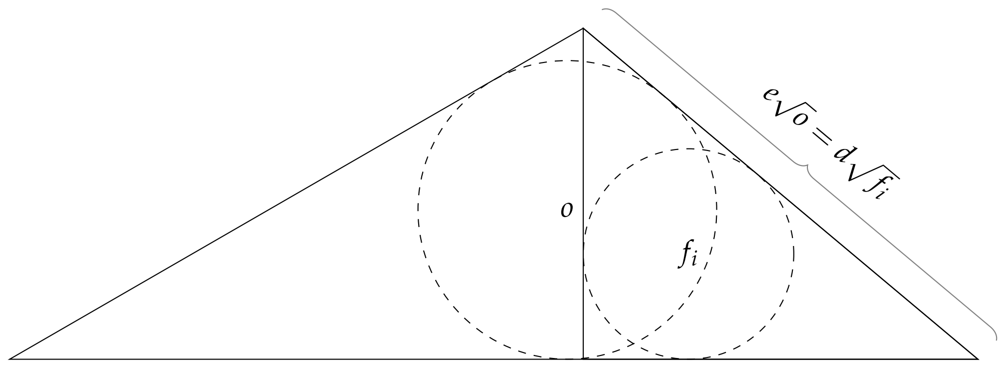


Figure 4.5: The ratios d and e are constant for all similar triangles.

Moving forward, in Figure 4.6, we display the situation when packing a hat into (without loss of generality) the right leg of the container. f_i is the relevant factor from the split key, a_i is the hat's incircle and b_i represents the hat's rounding.

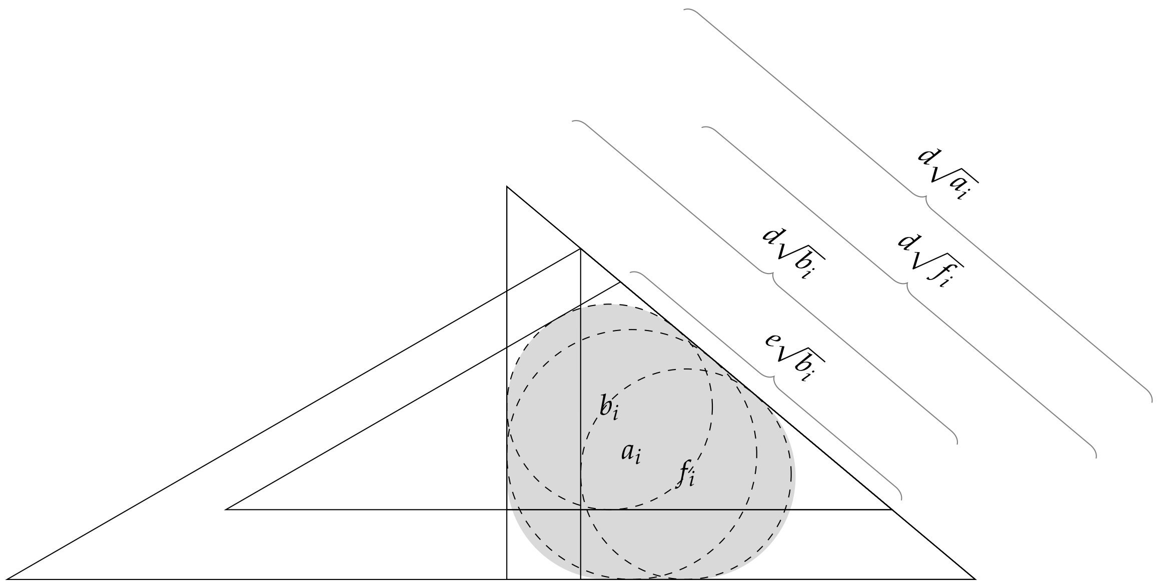


Figure 4.6: Various measurements when packing a rounded hat.

The hat is placed in such a way that it will never overlap the bottom or the right leg of the containing triangle, so it is sufficient to show that it does not overlap the left leg. We can tell from Figure 4.6 that this does not happen if the length of the right side of the triangle the hat is based on ($d\sqrt{a_i}$), minus the length of the right side of the $(b_i, 0)$-triangle similar to the containers right side ($d\sqrt{b_i}$), plus the length of the right side of the $(b_i, 0)$-triangle similar to the container ($e\sqrt{b_i}$) is at most the length of the container's right leg ($d\sqrt{f_i}$). So the following condition has to hold:

$$d\sqrt{a_i} - d\sqrt{b_i} + e\sqrt{b_i} \leq d\sqrt{f_i}$$

As previously observed, $e = d\sqrt{f_i/o}$:

$$d\sqrt{a_i} - d\sqrt{b_i} + d\sqrt{f_i/o}\sqrt{b_i} \leq d\sqrt{f_i}$$

In our case, the incircle of the triangle has exactly the maximal area which we want to pack, so $o = a$. But even if $o \geq a$, the inequality is true if

$$d\sqrt{a_i} - d\sqrt{b_i} + d\sqrt{f_i/a}\sqrt{b_i} \leq d\sqrt{f_i}$$

We can also divide by d and factor out $\sqrt{b_i}$ to get:

$$\sqrt{a_i} - (1 - \sqrt{f_i/a})\sqrt{b_i} \leq \sqrt{f_i} \tag{1}$$

Let j be the index of the other hat to be packed. We know (from the conjugatedness) that the sum of both hats' incircles does not exceed the total area a, so $a_i + a_j \leq a$. Also, $f_i + f_j \geq a$, as demonstrated in Figure 4.7: In right triangles, $f_1 + f_2$ is exactly a, because as its two halves are *similar* to the large triangle, the two halves' areas add up to the container triangle's area, and the ratio between the areas of a triangle and its incircle are constant. When making the upper angle more obtuse, but letting f_1 and f_2 stay the same, the incircle only shrinks (like the dotted variant in Figure 4.7).

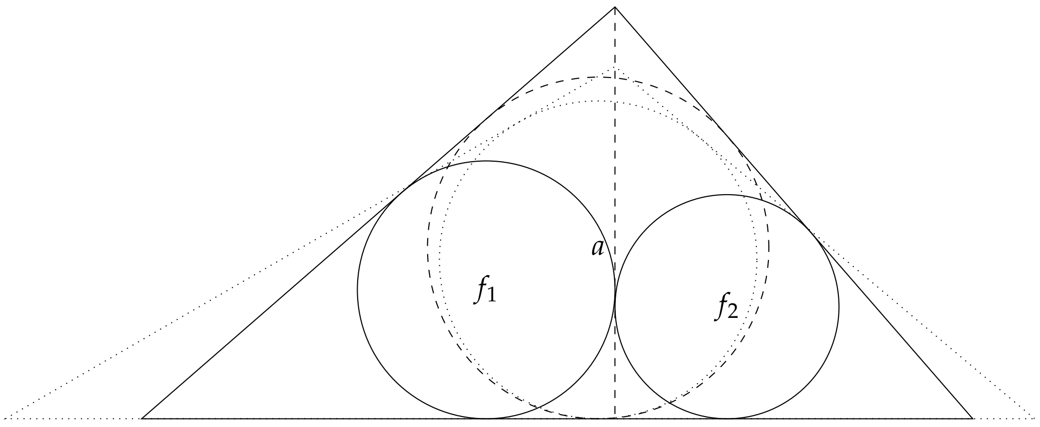


Figure 4.7: In non-acute triangles, $f_1 + f_2 \geq a$.

Putting it together, by Theorem 1, our hat it is rounded by

$$b_i \geq a_i - f_i\frac{a_j}{f_j} \geq a_i - f_i\frac{a - a_i}{a - f_i} = \frac{a_i(a - f_i) - f_i(a - a_i)}{a - f_i} = a\frac{a_i - f_i}{a - f_i}$$

Insert that into Equation (1):

$$\sqrt{a_i} - (1 - \sqrt{f_i/a})\sqrt{a\frac{a_i - f_i}{a - f_i}} \leq \sqrt{f_i}$$

Then bring the subtrahend to the right and square both sides (both are positive):

$$a_i \leq f_i + 2\sqrt{f_i}(1 - \sqrt{f_i/a})\sqrt{a\frac{a_i - f_i}{a - f_i}} + (1 - \sqrt{f_i/a})^2 a\frac{(a_i - f_i)}{a - f_i}$$

Subtract f_i and divide by $\sqrt{a_i - f_i}$:

$$\sqrt{a_i - f_i} \leq \frac{2\sqrt{f_i}(1 - \sqrt{f_i/a})\sqrt{a}}{\sqrt{a - f_i}} + (1 - \sqrt{f_i/a})^2 a\frac{\sqrt{a_i - f_i}}{a - f_i}$$

Rearrange:

$$\sqrt{a_i - f_i}\frac{(a - f_i) - (1 - \sqrt{f_i/a})^2 a}{a - f_i} \leq \frac{2\sqrt{f_i}(1 - \sqrt{f_i/a})\sqrt{a}}{\sqrt{a - f_i}}$$

Simplify:

$$\sqrt{a_i - f_i}\frac{2\sqrt{f_i a} - 2f_i}{a - f_i} \leq \frac{2\sqrt{f_i a} - 2f_i}{\sqrt{a - f_i}}$$

Then multiply with $\sqrt{a - f_i}$:

$$\sqrt{a_i - f_i}\frac{2\sqrt{f_i a} - 2f_i}{\sqrt{a - f_i}} \leq \frac{2\sqrt{f_i a} - 2f_i}{\sqrt{a - f_i}}\sqrt{a - f_i}$$

Finally, divide by the fraction to get:

$$\sqrt{a_i - f_i} \leq \sqrt{a - f_i} \iff a_i - f_i \leq a - f_i \iff a_i \leq a$$

From the conjugatedness we know that a_i is never larger than a, so Equation (1) is true and the hat always fits into the container. This completes the proof of Lemma 4. □

Until now, the container was always an $(a, 0)$-hat, which is essentially a non-rounded triangle with an incircle of a. The next lemma extends this idea to hats which are actually rounded. It is identical to Lemma 4, except that the rounding of the container hat is no longer 0, but b.

Lemma 5. Consider an (a, b)-hat with the associated split key $F = (f_1, f_2)$, and call its left- and right-angles α and β. For all (a, b, F)-conjugated tuples (a_1, b_1) and (a_2, b_2) with $a_1 + a_2 \leq a$, the following two shapes can be packed into the hat:

- a right (a_1, b_1)-hat with a right-angle of α and
- a right (a_2, b_2)-hat with a left-angle of β.

Proof. Lemma 4 tells us that this theorem is true for $b = 0$. Now, the container's corners can be rounded to the radius of a b-circle, and we need to show that the two hats from the previous construction still fit inside. But all of the two hat's corners are also rounded to (at least) the same radius (see Theorem 1), so they will never overlap the container, see Figure 4.8. □

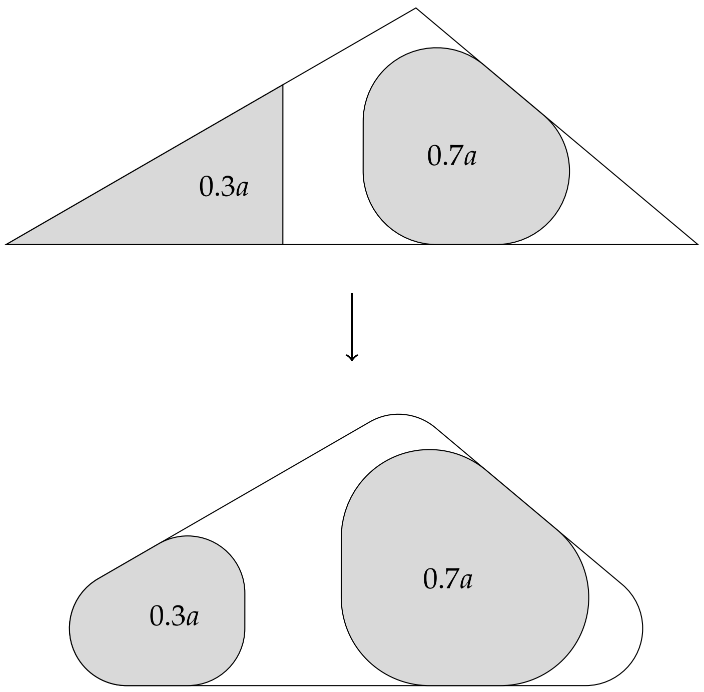


Figure 4.8: Rounding all hats' corners by the same radius does not affect the packing.

With these preparations, we can finally apply Split Packing to hats:

Theorem 5. *Given an (a, b)-hat, all circle instances with a combined area of at most a and a minimum circle size of at least b can be packed into that hat.*

Proof. We proof by induction that we can pack each $C \in \mathbb{C}(a, b)$ into the hat:

If C only consists of a single circle, it can be packed into the hat, as it is at most as big as the hat's incircle.

Now assume that for any $0 \leq b \leq a$, any (a, b)-hat could pack all circle instances into $\mathbb{C}(a, b)$ with at most n circles. Consider a circle instance $C \in \mathbb{C}(a, b)$ containing $n + 1$ circles. Definition 13 tells us how to compute the split key F. Then we know from Theorem 1 that Split will partition C into two subinstances $C_1 \in \mathbb{C}(a_1, b_1)$ and $C_2 \in \mathbb{C}(a_2, b_2)$, whose parameters are (a, b, F)-conjugated. As Split can never return an empty instance (except for $|C| = 1$, a case which we handled above), each subinstance will contain at most n circles. We know from Lemma 5 that, for all pairs of (a, b, F)-conjugated tuples, we can find two hats with matching parameters which fit into the container hat. By assumption, these hats can now pack all instances from $\mathbb{C}(a_1, b_1)$ and $\mathbb{C}(a_2, b_2)$, respectively, which means that they can especially also pack C_1 and C_2. If we then pack the two hats into the container, we have constructed a packing of C into the container hat.

By induction, we can pack each $C \in \mathbb{C}(a, b)$ into the (a, b)-hat. □

4.2 Non-acute triangles

As a non-acute triangles is only a special case of a hat, this section is will be pleasantly short.

Theorem 6. *Given a non-acute triangle with an incircle of area a, all circle instances with a combined area of up to a can be packed into the triangle, and this bound is tight. See Figure 4.9 for some example packings. Expressed algebraically, for a triangle with side lengths a, b, and c, the critical density is*

$$\sqrt{\frac{-(a-b-c)(a+b-c)(a-b+c)}{(a+b+c)^3}}\pi < 53.91\%.$$

Proof. The triangle is an $(a, 0)$-hat, which by Theorem 5 is a $\mathbb{C}(a)$-shape.

On the other hand, a single circle of area $a + \varepsilon$ cannot be packed, as the incircle is by definition the largest circle which fits into the triangle.

As for the algebraic formulation of the critical density, the area of the triangle can be calculated using Heron's formula:

$$\Delta(a,b,c) := \sqrt{s(s-a)(s-b)(s-c)} \text{ with } s = \frac{a+b+c}{2}$$

It is also known that the radius of the incircle of this triangle is

$$R(a,b,c) := \frac{\Delta(a,b,c)}{s} \text{ with } s = \frac{a+b+c}{2},$$

so the incircle has an area of

$$I(a,b,c) = \pi R(a,b,c)^2 = \frac{(a+b-c)(c+a-b)(b+c-a)}{4(a+b+c)}.$$

Finally, the ratio between the areas of the circle and the triangle can be calculated to be

$$\frac{I(a,b,c)}{\Delta(a,b,c)} = \sqrt{\frac{-(a-b-c)(a+b-c)(a-b+c)}{(a+b+c)^3}}\pi.$$

□

4.3 Isosceles triangles

Split Packing can pack with critical density not only into non-acute triangles, but also into isosceles triangles, as long as its two symmetric sides are not longer than the base. In this sense, it works for all triangles inbetween a right isosceles triangle and an equilateral triangle.

Definition 14. A *thick isosceles triangle* is an isosceles triangle whose legs are at most as long as its base, but whose top angle is at least $\frac{\pi}{4}$. Examples are shown in Figure 4.10.

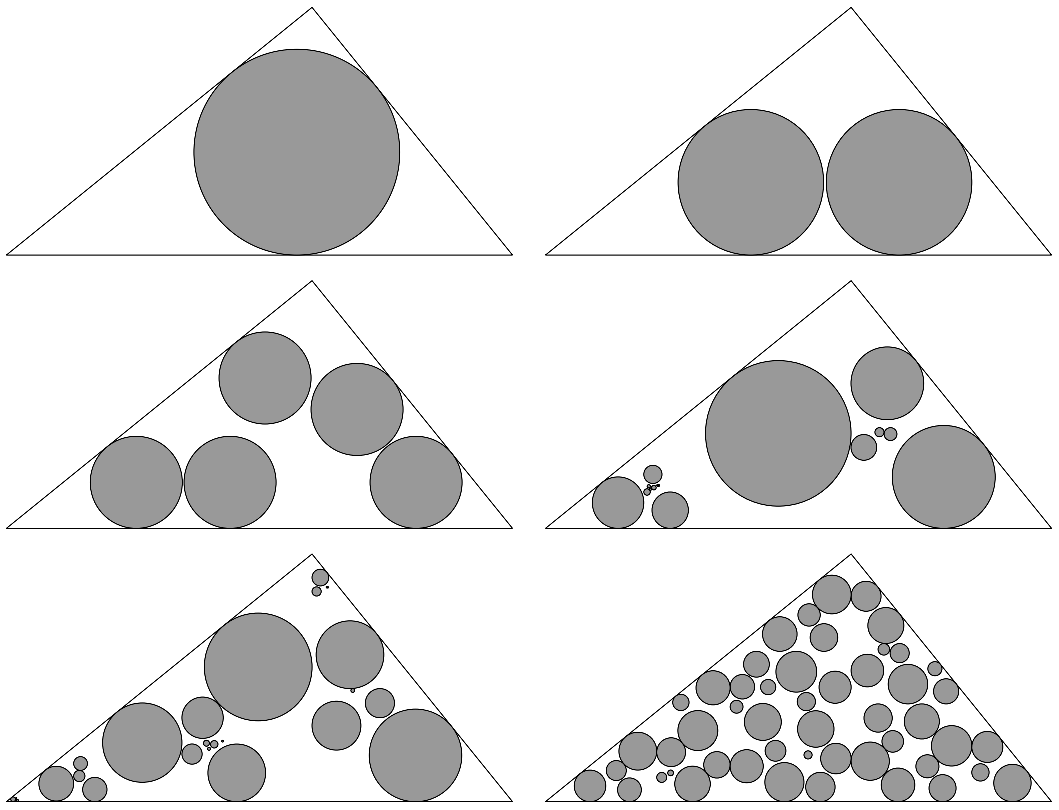


Figure 4.9: Example packings of various circle instances into a right triangle produced by Split Packing.

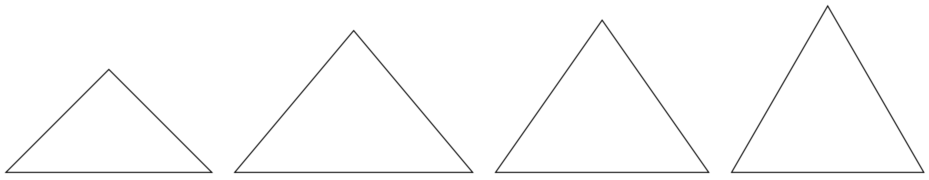


Figure 4.10: Range of considered isosceles triangles.

To prove the worst case for these triangles, we first make a statement about a triangle's *twincircles*:

Lemma 6. Two touching equal circles, placed in the corners connected by a triangle's longest side, are the triangle's twincircles, meaning there are no two larger equal circles which can be packed.

Proof. Call the circles' radius r. When eroding the triangle by r, the remaining area is a triangle similar to the original one, whose longest side has a length of $2r$, see Figure 4.11. This means that when eroding the triangle by $r + \varepsilon$, the longest side of the remaining area must be shorter than $2r$. But to pack two circles with radius $r + \varepsilon$, their centers need to be placed at least $r + \varepsilon$ away from the triangle's boundary and at least $2r + 2\varepsilon$ away from each other. The construction shows that this is not possible. □

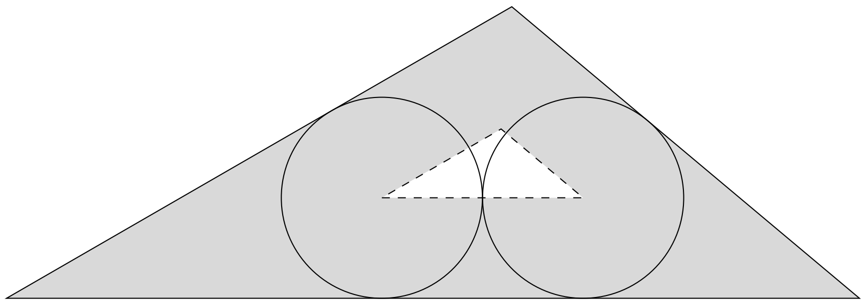


Figure 4.11: Two circles larger than these cannot be packed.

In the previous section, the target area a which we wanted to pack always was the area of the triangles incircle. For thick isosceles triangles, the twincircle area is *smaller* than the incircle area, which means it will not always be possible to pack the area of the incircle. In this section, we will argue that we can always pack the twincircle area into thick isosceles triangles.

Definition 15. The split key of a thick isosceles triangle is $(1, 1)$.

Lemma 7. Consider a thick isosceles triangle with a twincircle area of a and call its symmetric angles α. For all $(a, 0, (1, 1))$-conjugated tuples (a_1, b_1) and (a_2, b_2), the following two shapes can be packed into the triangle:

- a right (a_1, b_1)-hat with a right-angle of α and
- a right (a_2, b_2)-hat with a left-angle of α.

Proof. See Figure 4.13 for example packings for different a_1 and a_2.

First, we show that the hats do not intersect each other: In any isosceles triangle, the projection factors for both sides, p_1 and p_2 are equal. So in Figure 4.12, we construct the width of the triangle's base to be $2\sqrt{\frac{a}{2}}p_1 = \sqrt{2ap_1^2} = \sqrt{a(p_1^2 + p_2^2)}$. By Lemma 3, this means that the hats do not overlap.

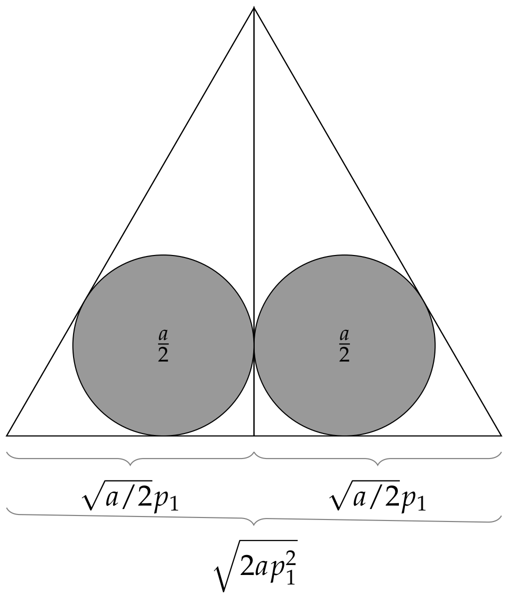


Figure 4.12: The width of an isosceles triangle is $\sqrt{2ap_1^2}$.

Second, we show that the hats do not intersect the container. We are going to re-use the proof from Lemma 4 here. On page 29, we showed that the hats do not overlap the container if certain conditions are met. These conditions are also true here:

- $a_1 + a_2 \leq a$, the hat's incircles add up to at most a because of the conjugatedness of the tuples.
- $f_i + f_j \geq a$, in this case, the incircle-areas of the triangles sides are equal to a by assumption.
- $o \geq a$, meaning that the triangles incircle is at least as large as the area of the twincircles. This can be demonstrated like in Figure 4.7 on page 29, but in the reverse direction: For a right isosceles triangle, the twincircle area a is exactly equal to the triangle's incircle. Leaving the twincircles in place, but increasing the top angle, the incircle only gets larger.

Thus, also in this case, the hats do not overlap the container. This concludes the proof. □

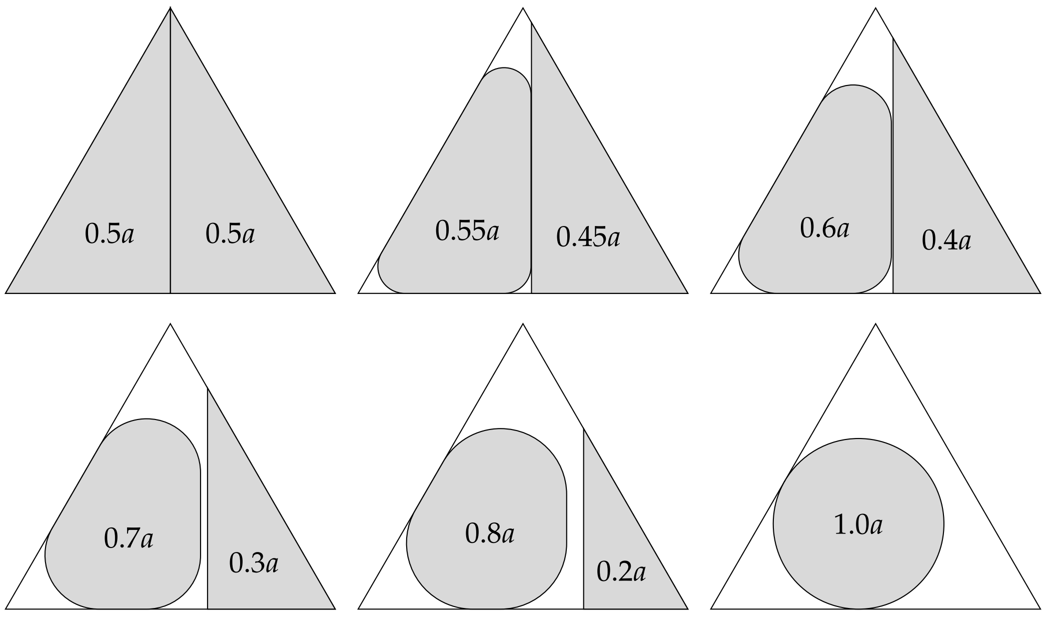


Figure 4.13: Packing two hats into an equilateral triangle.

Theorem 7. *Given a thick isosceles triangle with a twincircle area of a, all circle instances with a combined area of up to a can be packed into the triangle, and this area bound is tight. See Figure 4.14 for some example packings. Expressed algebraically, for a triangle with legs of length b and a base of length c, the critical density is*

$$48.60\% < \frac{(c - 2b + \sqrt{4b^2 - c^2})^2 \pi}{2c\sqrt{4b^2 - c^2}} < 53.91\%$$

Proof. By Lemma 7 and the Split Packing Theorem, the triangle is a $\mathbb{C}(a)$-shape.

In Lemma 6, we argued that any two equal circles larger than the twincircles cannot be packed.

We use Heron's formula again to calculate a critical density of

$$\frac{2I(\frac{c}{2}, b, h)}{\frac{c}{2}h} = \frac{(c - 2b + \sqrt{4b^2 - c^2})^2 \pi}{2c\sqrt{4b^2 - c^2}}.$$

□

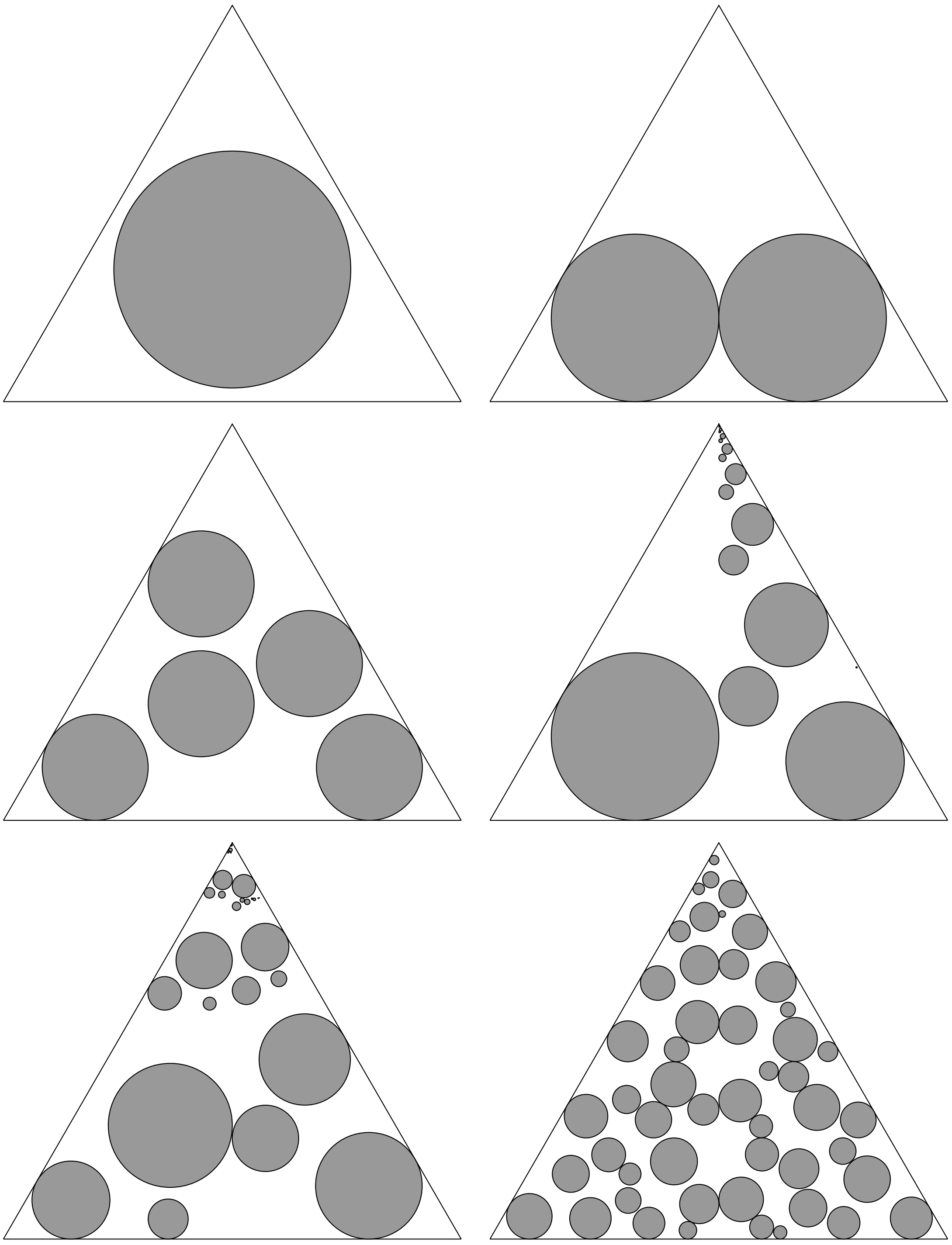


Figure 4.14: Example packings of various circle instances into an equilateral triangle produced by Split Packing.

4.4 The problem with acute triangles

A class of triangles for which we could not prove the critical density are general acute triangles. The problem is that the condition for Lemma 3 is not met, which means that the two hats may overlap:

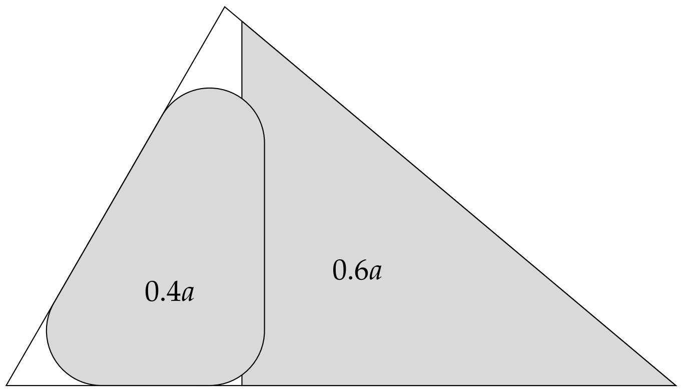


Figure 4.15: For general acute triangles, the two hats may overlap.

We worked under the following assumption:

Conjecture 1. A circle instance can be packed into a triangle if the circles' combined area does not exceed the triangles incircle or twincircle, whichever is smaller.

If this conjecture is true, surely there are strategies which can pack into acute triangles with critical density. For example, we attempted to split the circle instance into four subinstances using a slightly modified Split algorithm, and then to pack four hats into the container, like in Figure 4.16. Again, this is motivated by the observation that, when splitting each circle top-down into four equal circles, this strategy always works because the triangle is recursively divided into four similar triangles.

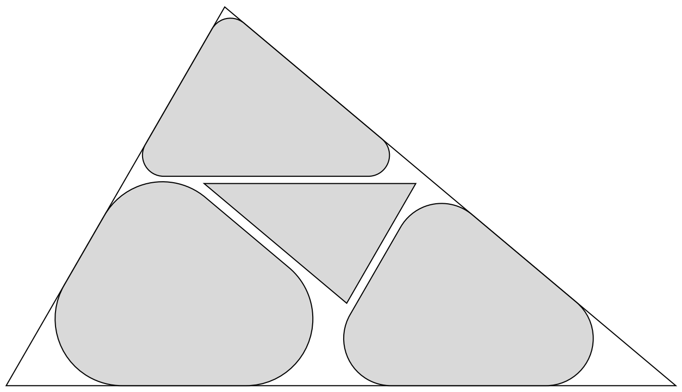


Figure 4.16: Packing four hats into an acute triangle.

Unfortunately, this strategy fails for some instances, as depicted in Figure 4.17. For this instance, the largest group, consisting of a single circle, cannot be packed if *any* of the smaller groups is packed into the top or the left corner, because the remaining free space is not wide enough for the circle. Other strategies or a case distinction would need to be considered.

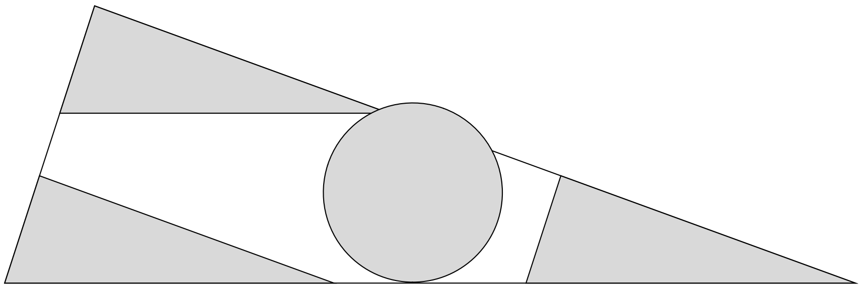


Figure 4.17: This strategy does not work for a container with a right-angle of $\frac{\pi}{10}$, incircle 1 and the circle instance $\{0.55, 0.15, 0.15, 0.15\}$.

4.5 Rectangles

Some of our main results concern the packing of circles into a square. We will cover that packing problem in the next chapter, along with some generalizations on the objects which are to be packed, which is why we do not repeat it here. Instead, we first turn toward rectangles.

Split Packing can be used to pack circles with critical density into rectangles. The basic idea is to divide the rectangle along a diagonal, and pack the subinstances produced by Split into these two halves. If one group gets larger than half, it gets rounded enough so that it still works.

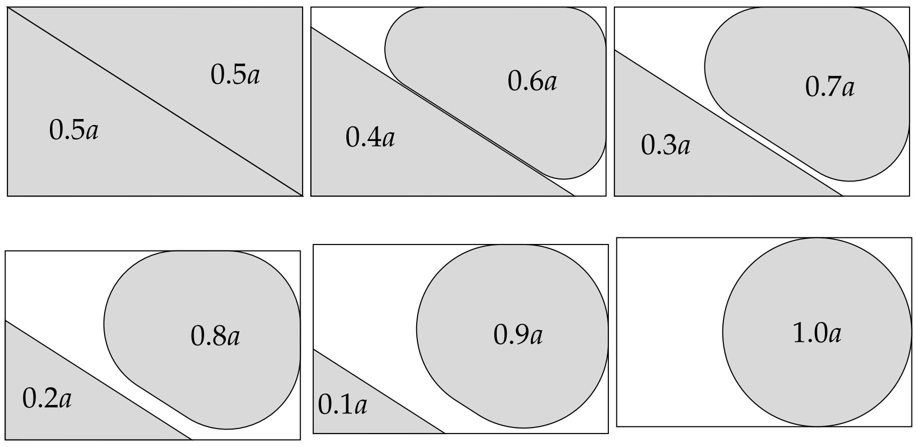


Figure 4.18: Hat-in-rectangle packings for different values of a_1 and a_2.

At this point, we do not know how to handle *all* rectangles, and this has to do with the different classes of worst-cases which appear when packing circles into rectangles of different ratios. Figure 4.19 shows several examples. At the top, we show the conjectured worst-case instance. For squares, we already discussed the worst case—it consists of the square's twincircles, the two largest circles which can be packed into the square.

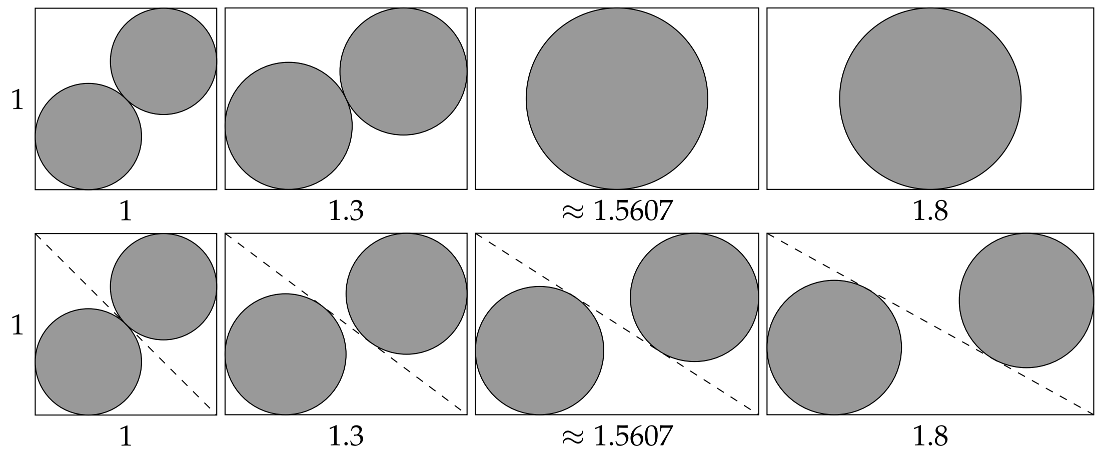


Figure 4.19: Rectangles of different aspect ratios. Top: Their conjectured worst-case instances. Bottom: The incircles of their halves. Aspect ratios between 1 and ≈ 1.5607 are the only ones where the area of the worst-case instance is larger than the area of the two circles inscribed in the halves.

For rectangles with a slightly larger aspect ratio (like 1:1.3), we conjecture that the worst-case instance still consists of the rectangle's twincircles, but we do not know how to pack this area using Split Packing, because when cutting the rectangle along the circles' tangent, the resulting pieces are not triangular. At this point, we do not know how to deal with these kinds of quadrilaterals recursively.

As the aspect ratio gets even larger, at some point the twincircle area gets larger than the incircle of the rectangle, whose diameter is as long as the rectangle's smaller side. Obviously, any circle larger than this incircle cannot be packed, so from this point forward, the worst-case instance of these "long" rectangles will be their incircles.

At the bottom of Figure 4.19, we divided all rectangles along their diagonal, and inscribed circles into the resulting halves. We will show that we can pack circles with critical density into a rectangle if the area of those two circles is at least as large as the worst-case area: We know from the previous sections that we can pack right hats with incircle density. It is left to show that, if one of the two hats should get larger than the other one, they both can still be packed.

What is the aspect ratio of the rectangle where the combined area of the two circles incribed in the halves is exactly equal to the incircle area? In Figure 4.20, we see that the area of the incircle is $A = \pi(\frac{h}{2})^2$.

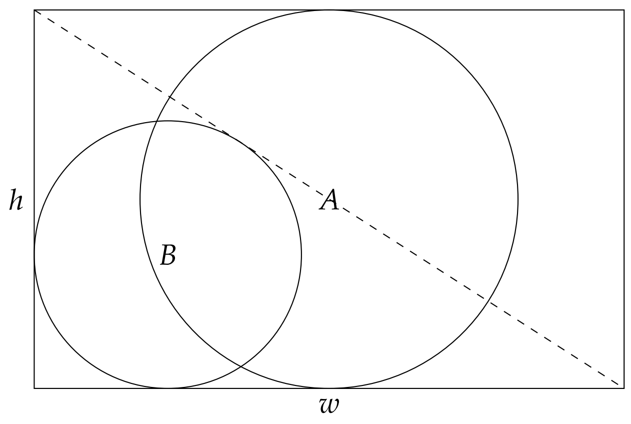


Figure 4.20: Constructing the case where $A = 2B$.

The proof of Theorem 6 on page 32 discusses how to use Heron's formula to algebraically find the area of a triangle, given its side lengths. In our case, $A = 2B$ is equivalent to

$$\frac{\pi}{4}h^2 = 2I(h, w, \sqrt{h^2 + w^2}) = \frac{\pi a w(h + w - \sqrt{h^2 + w^2}}{h + w + \sqrt{h^2 + w^2}}$$

and this equation this is true for $w = \frac{2+3\sqrt{2}}{4}h \approx 1.5607h$. This means that all rectangles whose side ratio is at least this large can be packed with critical density, simply by using the split key $F = (1, 1)$ and by placing two right hats in the rectangle.

Lemma 8. Consider a rectangle whose longer side w is at least $\frac{2+3\sqrt{2}}{4} \approx 1.5607$ times as long as its shorter side h and let a be the area of its incircle. Any two $(a, 0, (1, 1))$-conjugated right hats similar to the two halves of the rectangle when cutting diagonally from corner to corner can be packed into the rectangle.

Proof. See Figure 4.18 for some example packings. The proof works analogously to the proof of Lemma 4. □

Theorem 8. *Given a rectangle whose longer side w is at least $\frac{2+3\sqrt{2}}{4} \approx 1.5607$ times as long as its shorter side h, let a be the area of its incircle. All circle instances with a combined area of up to a can be packed into the rectangle, and this area bound is tight. See Figure 4.21 for some example packings. Expressed algebraically, the critical density is $\frac{\pi h}{4w}$.*

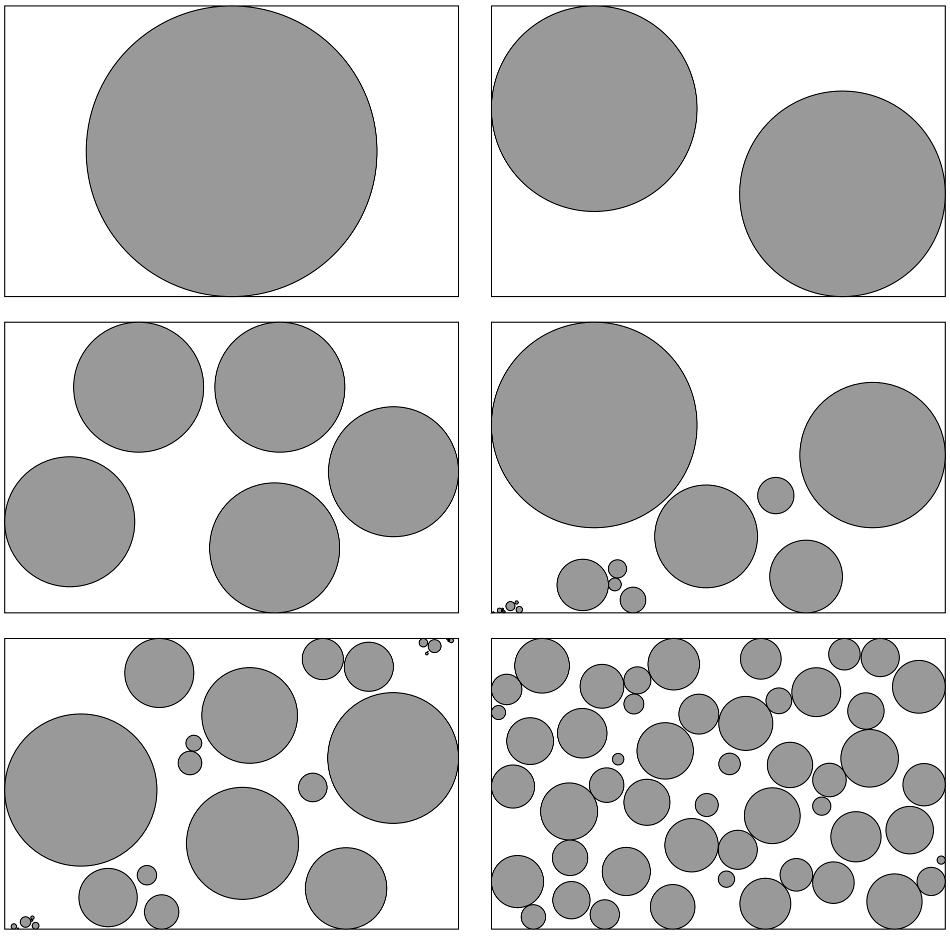


Figure 4.21: Example packings of various circle instances in a rectangle with aspect ratio 1:1.57 produced by Split Packing.

Proof. By Lemma 8 and the Split Packing Theorem, the rectangle is a $\mathbb{C}(a)$-shape.

The a-circle's diameter is equal to the rectangle's smaller side, so it is clear that any larger circles cannot be packed.

The critical density is the ratio of the circle's and the rectangle's area, which can be calculated to be

$$\frac{\pi(\frac{h}{2})^2}{wh} = \frac{\pi h}{4w} < 50.33\%.$$

□

5 Packing rubies

In the previous chapter, we showed how Split Packing can be used to pack circular objects into triangles and rectangles. As we explored generalizations of Split Packing during the creation of this thesis, we usually tried to find additional container shapes for which we could use Split Packing to pack circles with up to critical density.

But at one point, we decided to try whether we could vary the “other half” of packing problems: The objects. We rarely had thought of packing objects other than circles.[1] We were surprised to find out that, when we allowed to rotate objects, we could also use Split Packing to pack squares into a square container with up to critical density! Shortly after, we discovered that we could also use it to pack octagons with up to critical density. Both modified packing problems could be solved with only a very small modification of the algorithm.

As we know now, the reason why we can pack circles, squares, as well as octagons, is that all these shapes are subsets of the very special shape shown in Figure 5.1. We call this shape *ruby*, because it looks like a cut gemstone from the side:

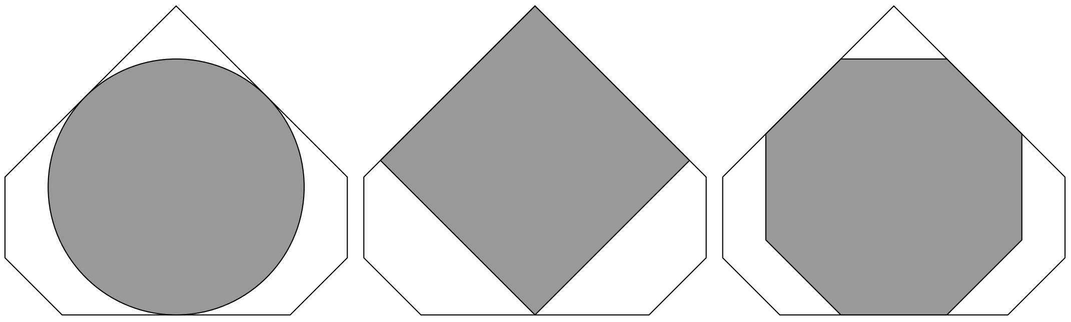


Figure 5.1: The “ruby” shape and its contained circle, square and octagon.

In this chapter, we will show how to pack rubies into squares and isosceles right triangles with up to critical density. As corollaries, we get critical-density packing algorithms for circles, squares, and octagons for these containers for free! In fact it turns out that, if you are given a set of objects, no matter what their shape is, if you can enclose them in rubies so that the combined area of the rubies does not exceed the critical density, then you can pack the shapes into these containers.

[1]After all, the preliminary title of this thesis was *Circle Packing Algorithms*!

5.1 Gems and rubies

For this generalization, we need another new shape, which has a similar function as previous chapter's *hats*. Our ruby shape has a very distinct way its corners are cut off. We generalize this cutoff to get to a family of shapes which we call *gems*[2]:

Definition 16. For each $0 \le b \le a$, an (a, b)*-gem* is an isosceles right triangle with an incircle of area a, whose acute corners are cut off like shown in Figure 5.2: Each cut forms an angle of $\frac{3}{4}\pi$ with the triangle's boundary, and the distance of each cut to the triangle's original corner is

$$c(b) := \frac{(2+\sqrt{2})(2\sqrt{\sqrt{2}-1}-1)}{\sqrt{2(\sqrt{2}-1)\pi}}\sqrt{b} \approx 0.6078\sqrt{b}.$$

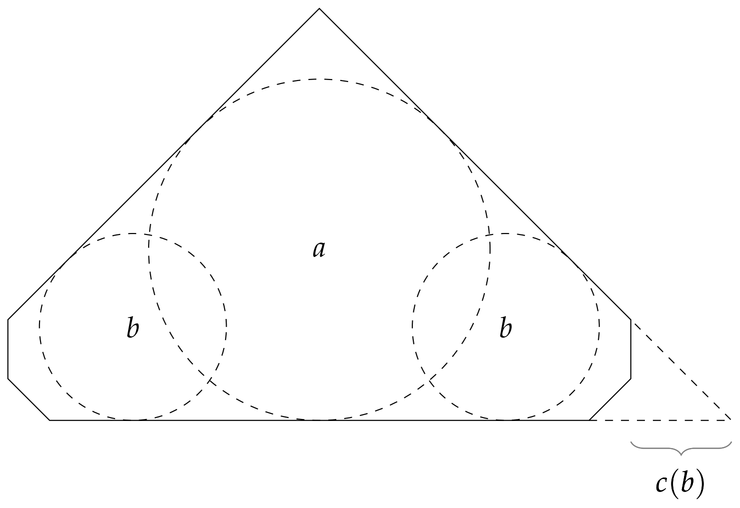


Figure 5.2: An (a, b)-gem.

Definition 17. A gem's associated split key is $(1, 1)$, as usual for symmetric shapes.

For the following proofs, we need to know a gem's dimensions in detail. Consider an (a, b)-gem and let $r = \sqrt{\frac{a}{\pi}}$ be the radius of its incircle. By construction in Figure 5.3, the hat has

- cut-width $c(b) = \frac{(2+\sqrt{2})(2\sqrt{\sqrt{2}-1}-1)}{\sqrt{2(\sqrt{2}-1)\pi}}\sqrt{b}$ (by definition),
- diagonal $D(a) = \sqrt{2}(r\sqrt{2} + r) = (2 + \sqrt{2})\sqrt{\frac{a}{\pi}}$,
- width $w(a, b) = 2(r\sqrt{2} + r) - 2c(b) = (2 + 2\sqrt{2})\sqrt{\frac{a}{\pi}} - 2c(b)$, and
- corner-width $W(a, b) = (2 + 2\sqrt{2})\sqrt{\frac{a}{\pi}} - c(b)$.

[2]A ruby is a special gem, get it?

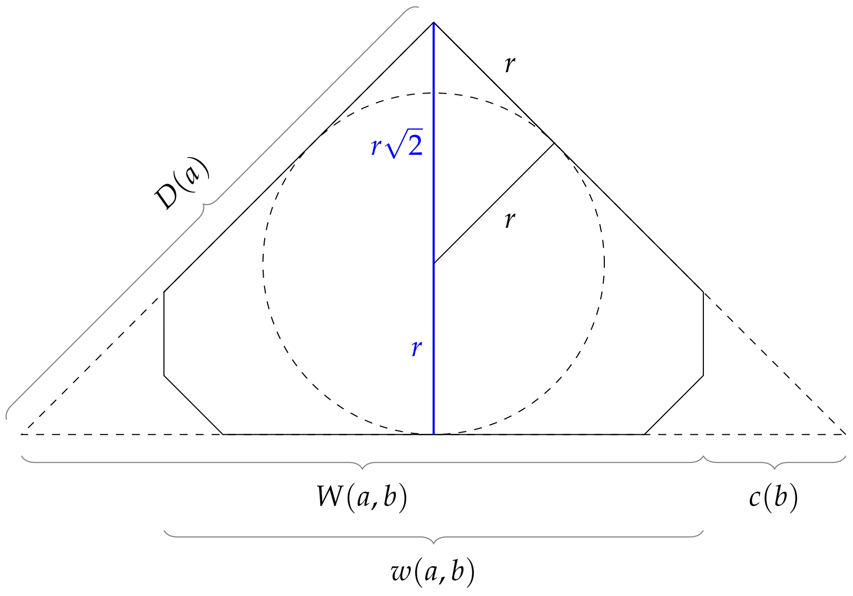


Figure 5.3: Dimensions of an (a,b)-gem.

The definition of a gem is designed so that we get our ruby shape exactly for $a = b$:

Definition 18. We call an (a,a)-gem an a-*ruby*.

An a-ruby contains the following shapes, see Figure 5.2:

- a circle of area a,
- a square of area $\frac{3+2\sqrt{2}}{2\pi}a \approx 0.9276a$, and
- a regular octagon of area $\frac{8\sqrt{2}-8}{\pi}a \approx 1.0548a$.

This is because if the circle's radius is $r = \sqrt{\frac{a}{\pi}}$, then the square has an edge length of $(1 + \frac{1}{\sqrt{2}})r$, so its area is $(1 + \sqrt{2} + \frac{1}{2})r^2$. Regular n-gons with an apothem[3] of r have an area of $nr^2 \tan(\frac{\pi}{n})$, which, for $n = 8$, amounts to $8r^2(\sqrt{2} - 1)$.

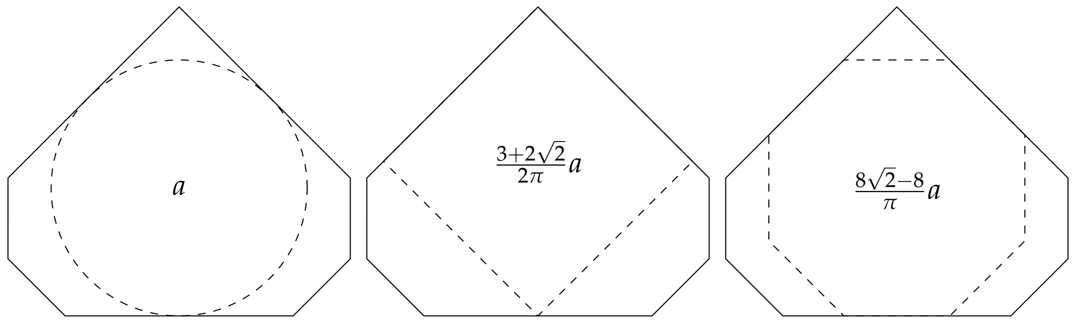


Figure 5.4: An a-ruby and its contained circle, square and octagon.

[3]The apothem is the distance of the polygon's center to its sides.

A ruby itself can be calculated to have an area of $\frac{8\sqrt{2(7+5\sqrt{2})}-3(7+4\sqrt{2})}{\pi}a \approx 1.4225a$.

We will also use a modified version of gems and rubies for which only one corner is cut off. We will call these gems *sharp*:

Definition 19. A *sharp* (a,b)*-gem* is an isosceles triangle where only one corner is cut off like in the definition of a gem, see Figure 5.5. A *sharp ruby* is a sharp (a,a)-gem.

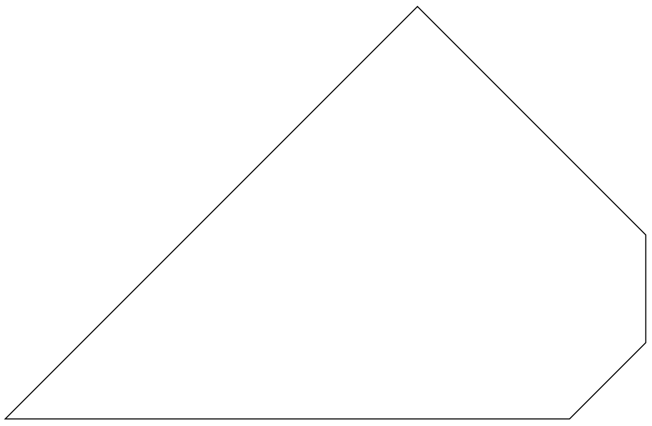


Figure 5.5: A *sharp* ruby.

A sharp ruby is a superset of a regular ruby with the same parameters and can be calculated to have an area of $\frac{4\sqrt{2(7+5\sqrt{2})}-5\sqrt{2}-9}{\pi}a \approx 1.6388a$.

From this point forward, we will no longer pack circles, but (regular and sharp) rubies. To simplify talking about these instances, we make the following definition:

Definition 20. A *ruby instance* is defined excactly like a circle instance (see Definition 4), except that the real numbers now define the areas of the rubies' incircles. $\mathbb{R}$ is the set of all ruby instances, $\mathbb{R}(a,b)$ is defined analogously. Similarly, we call the set of all *sharp ruby instances* $\mathbb{S}$, and define $\mathbb{S}(a,b)$ analogously.

One way to think about ruby instances is that they consist of the rubies "built around" the circles of a circle instance.

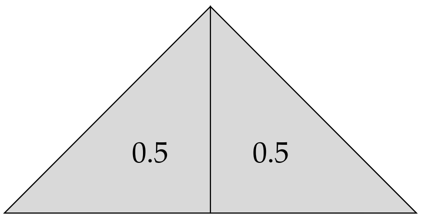
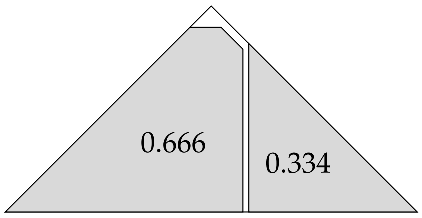
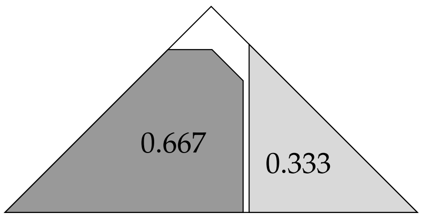
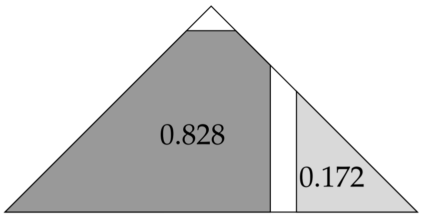
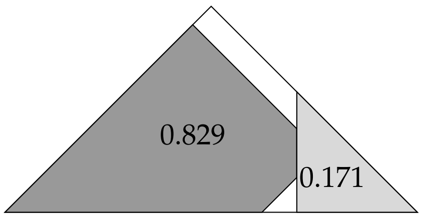
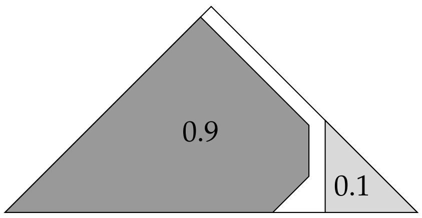
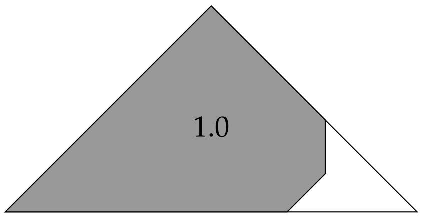
Lemma 9. Any two $(a,0,(1,1))$-conjugated sharp gems can be packed into a sharp $(a,0)$-gem.

Proof. Assume without loss of generality that the two gems are a sharp (a_1,b_1)-gem and a sharp (a_2,b_2)-gem with $a_1 \geq a_2$, meaning that we always swap the larger gem to the left. Also, let us normalize a to 1 to make calculations easier. The proofs hold for all a, one just needs to multiply all areas with a and all lengths with $\sqrt{a}$.

Again, we will show that we can always place the gems so that (1) they do not overlap each other and (2) they do not overlap the container. See Table 5.1 for an overview of the different cases, along with some more informal reasoning.

The smaller gem is always placed so that its tip touches the bottom of the container gem and its left corner fits into the containers right corner. By assumption, its incircle is smaller than 1/2, so it

Table 5.1: Gem-in-gem packings for different values of a_1 and a_2.

	The two gems are placed so that their tips touch the container's base and point toward each other. We assume $a_1 \geq \frac{1}{2}$ and normalize $a_1 + a_2$ to 1.
	They are then pushed as far to the left/right as possible. As a_1 grows, the gem becomes "cut-off" enough so that it does not overlap the container.
	As soon as $a_1 > \frac{2}{3}$, we know from Theorem 1 that $b_1 = a_1$, so the gem takes the form of a sharp ruby from this point forward. We display it in a darker shade here.
	When $a_1 = 2\sqrt{2} - 2$, the ruby touches the container's right leg.
	So when the ruby gets larger than that, it has to be rotated so that its tip points up. That these two positions exactly fit is the defining property of the ruby shape.
	In this orientation, it can then grow without overlapping the container and the smaller gem.
	When $a_1 = 1$, the ruby touches all sides of the container triangle.

will fit inside the right half of the container and thus never overlap the container's boundary. For the larger gem, there are three cases:

Case 1 When $a_1 \leq \frac{2}{3}$, we know from Theorem 1 that $b_1 \geq 2a_1 - 1$. We have to show that the corner-width of our gem does not exceed the diagonal of the container:

$$W(a_1, 2a_1 - 1) \leq D(1)$$

$$\iff (2+2\sqrt{2})\sqrt{\frac{a_1}{\pi}} - \frac{(2+\sqrt{2})(2\sqrt{\sqrt{2}-1}-1)}{\sqrt{2(\sqrt{2}-1)\pi}}\sqrt{2a_1 - 1} \leq \frac{2+\sqrt{2}}{\sqrt{\pi}}$$

Expressions of the form $f\sqrt{x} - g\sqrt{2x-1}$ have their only extremum at $x = \frac{f^2}{2(f^2-2g^2)}$. This point turns out to be a global minimum for $f, g > 0$. In our case, this minimum point is at $a_1 = \frac{1+5\sqrt{2}+4\sqrt{1+5\sqrt{2}}}{35} \approx 0.5553$. As we can check our inequality to be true for $a_1 = \frac{1}{2}$ as well as for $a_1 = \frac{2}{3}$, this means that it always holds between those two values.

Case 2 In the range $\frac{2}{3} \leq a_1 \leq 2\sqrt{2} - 2$, we know from Theorem 1 that $b_1 = a_1$, essentially making the gem into sharp a ruby. We have to show that the corner-width of the ruby is always smaller than the diagonal of the container:

$$W(a_1, a_1) \leq D(1)$$

But in the range for a_1 specified above, this is true:

$$W(a_1, a_1) = \frac{(1+\sqrt{2})^{\frac{3}{2}}}{\sqrt{\pi}}\sqrt{a_1} \leq \frac{(1+\sqrt{2})^{\frac{3}{2}}}{\sqrt{\pi}}\sqrt{2\sqrt{2}-2} = \frac{2+\sqrt{2}}{\sqrt{\pi}} = D(1)$$

In both Case 1 and Case 2, the two gems do not overlap each other because the projections of their incircles do not overlap. See Lemma 7 for a proof.

Case 3 For $a_1 = 2\sqrt{2} - 2$, the ruby touches the containers right leg. When $2\sqrt{2} - 2 < a_1 \leq 1$, it is no longer possible to place the gem like before without overlapping the container. In this case, we rotate it so that its bottom touches the bottom of the container gem and push it as far left as possible.

This is actually the defining property for the shape of the ruby: The cut-off is exactly so large that, when the growing ruby touches the container, the remaining space left over by the smaller gem is a (symmetric) kite, so that the ruby can be rotated.

In this orientation, it will never overlap the container: It can be enclosed in a (non-cut) $(a_1, 0)$-gem, with the same rotation as the container gem, and $a_1 \leq 1$, so our ruby always fits inside the container gem.

To show that the ruby does not overlap the smaller gem, we have to show that the following inequality holds for $2\sqrt{2} - 2 \leq a_1 \leq 1$: The corner-width of the ruby, added to the diagonal of the small gem, must not exceed the container's width:

$$W(a_1, a_1) + D(1 - a_1) \le w(1, 0)$$

$$\iff (2+2\sqrt{2})\sqrt{\frac{a_1}{\pi}} - \frac{(2+\sqrt{2})(2\sqrt{\sqrt{2}-1}-1)}{\sqrt{2(\sqrt{2}-1)\pi}}\sqrt{a_1} + \frac{2+\sqrt{2}}{\sqrt{\pi}}\sqrt{1-a_1} \le \frac{2\sqrt{2}+2}{\sqrt{\pi}}$$

$$\iff \frac{(1+\sqrt{2})^{\frac{3}{2}}}{\sqrt{\pi}}\sqrt{a_1} + \frac{2+\sqrt{2}}{\sqrt{\pi}}\sqrt{1-a_1} \le \frac{2\sqrt{2}+2}{\sqrt{\pi}}$$

Expressions of the form $f\sqrt{x} + g\sqrt{1-x}$ have their only extremum at $x = \frac{f^2}{f^2+g^2}$. This point turns out to be a global maximum for $f, g > 0$. In our case, the maximum is at $\frac{1+2\sqrt{2}}{7} \approx 0.5469$, and we have equality for $a_1 = 2\sqrt{2} - 2 \approx 0.8284$. If a_1 gets larger than that, the left hand side of the equation will only get smaller, so that the inequality holds. This concludes the proof of Lemma 9. □

Like with hats, "rounding" the container gem to a sharp gem with larger cut-off does not affect the packing properties:

Lemma 10. Any two $(a, b, (1, 1))$-conjugated sharp gems can be packed into a sharp (a, b)-gem.

Proof. Lemma 9 tells us that this theorem is true for $b = 0$. For $b > 0$, we flip the containers cut-off corner to the right, and also flip the smaller gem's cut-off corner so that it points to the right. Because the cut-off has the same shape, the inner gems will not overlap the container. □

Theorem 9. *Any sharp ruby instance $S \in \mathbb{S}(a, b)$ can be packed into a sharp (a, b)-gem.*

Proof. By Lemma 7 and the Split Packing Theorem. □

As a minor corollary, since rubies are subsets of sharp rubies of the same incircle, we can state the same property for regular rubies.

Theorem 10. *Any ruby instance $R \in \mathbb{R}(a, b)$ can be packed into an (a, b)-gem.*

Proof. This follows analogously to the reasoning with sharp gems. In this case, when cutting off the corners of the container gem, all corners of the contained gems are cut off by the same value, so they still fit inside. □

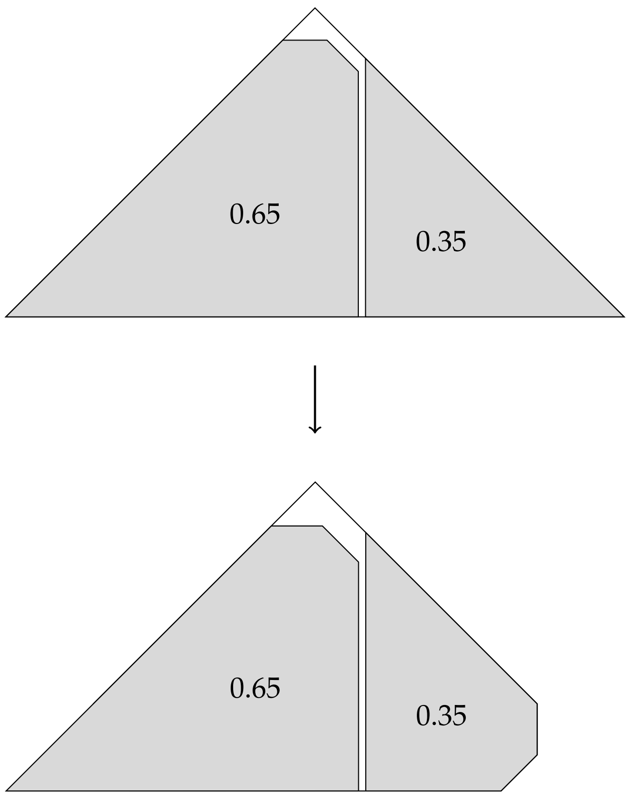


Figure 5.6: Cutting one corner of each gem by the same value does not affect the packing.

5.2 Isosceles right triangles

As isosceles right triangles are simply non-rounded sharp gems, this is again going to be a short section.

Theorem 11. *Given an isosceles right triangle with an incircle of area a, all sharp ruby instances with a combined incircle area of up to a can be packed into the triangle, and this area bound is tight. See Figure 5.7 for some example packings. Expressed algebraically, the critical density is*

$$4\sqrt{2(\sqrt{2}-1)} + 3\sqrt{2} - 7 \approx 88.34\%.$$

Proof. The triangle is a sharp $(a, 0)$-gem, and by Theorem 10 we can pack any sharp ruby instance $S \in \mathbb{S}(a)$ into such a gem.

A sharp ruby with an incircle area of more than a cannot be packed, as its incircle would be larger than the triangle's incircle.

The critical density can be calculated using the area of sharp rubies mentioned above. □

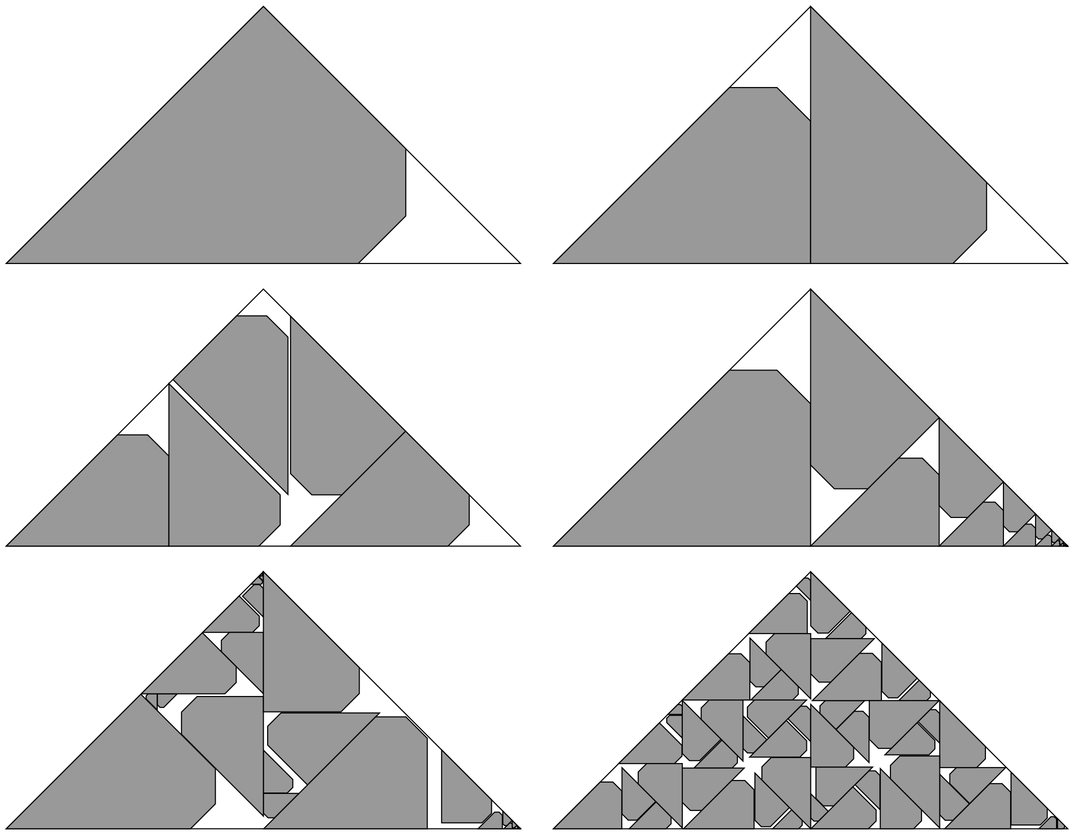


Figure 5.7: Example packings of various sharp ruby instances in an isosceles right triangle produced by Split Packing.

We can make the following observation here: Given an isosceles right triangle with an incircle area of a, and some arbitrary objects, if one can enclose each object in a sharp ruby, and the combined incircle area of these rubies is not larger than a, then the objects can be packed into the triangle. This is a very general result which can be used to directly solve some other interesting packing problems:

Corollary 1. *Given a right isosceles triangle with an incircle area of a, the following object instances can be packed into the triangle:*

- *all circle instances with a combined area of up to a,*
- *all octagon instances with a combined area of up to $\frac{8\sqrt{2}-8}{\pi}a \approx 1.0548a$, and*
- *all square instances with a combined area of up to $\frac{3+2\sqrt{2}}{2\pi}a \approx 0.9276a$, which is also exactly half of the triangle's area.*

All of these area bounds are tight. See Figure 5.8 for some example packings.

Proof. As demonstrated above, the three shapes are subsets of rubies in these specific area ratios. To pack them, enclose all of them in the smallest possible sharp ruby. The combined incircle area of these rubies will now be at most a. We then know from Theorem 7 that we can always pack these rubies.

The tightness of these bounds is obvious for circles and octagon. For squares, the tightness follows from the result by Göbel [15] that the largest two squares that can be packed (with rotation) into a square are quarter-squares. This implies that, when cutting the square along its diagonal, there is no larger square that can be packed into an isosceles right triangle than one with half the triangle's area. □

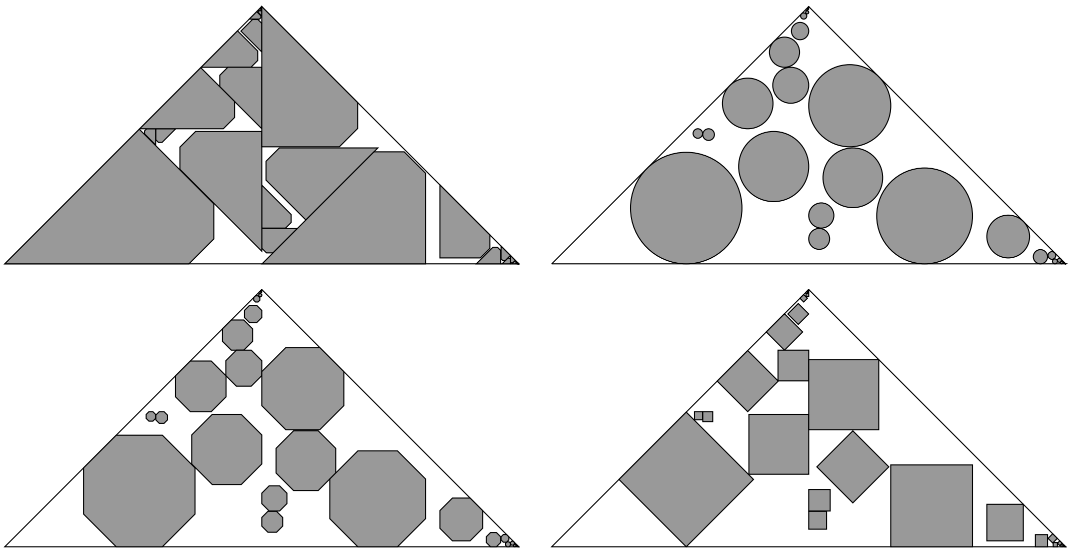


Figure 5.8: Example packings of ruby-, circle-, octagon-, and square instances in an isosceles right triangle produced by Split Packing.

5.3 Squares

Finally, we turn to square containers. Having established (a, b)-gems as $\mathbb{R}(a, b)$-shapes, to argue about the packing properties of squares is going to be relatively straightforward. We first argue about the worst-case instance for squares:

Lemma 11. Two touching equal circles, packed into opposing corners of a square, are the squares *twincircles*, meaning that there are no two larger equal circles which can be packed.

Proof. This can be proven similarly like in Lemma 6: Let r be the radius of these circles. When eroding the square by r, the result is a square with a diagonal of $2r$. When eroding by a larger radius $r + \varepsilon$, the diagonal will be smaller than $2r$. But the centers of the two circles need to be placed at least $2r + 2\varepsilon$ away from each other, and additionally need a distance of at least $r + \varepsilon$ from the square's boundary. We see that both constraints cannot be satisfied at the same time. □

Lemma 12. The twincircles of a square with area a have a combined area of

$$\frac{\pi}{3 + 2\sqrt{2}} a \approx 0.5390a.$$

Proof. We can construct the twincircles' radius r as seen in Figure 5.9:

$$2r + 2\frac{r}{\sqrt{2}} = \sqrt{a} \iff r = \frac{\sqrt{a}}{2+\sqrt{2}}$$

So, the combined area of the twincircles is

$$2\pi r^2 = 2\pi\frac{a}{4+4\sqrt{2}+2} = \frac{\pi}{3+2\sqrt{2}}a.$$

□

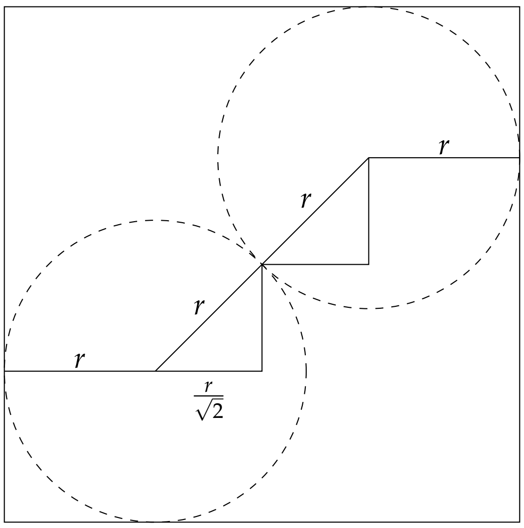


Figure 5.9: Constructing the twincircles' radius r.

Note that in the next lemma, we no longer pack *sharp* gems, but regular gems, whose two corners are cut off. The reason it will become clear in Figure 5.10: A large sharp $(a, 0)$-ruby cannot be packed into the square.

Lemma 13. Any two $(a, 0, (1, 1))$-conjugated gems can be packed into a square with twincircle area a.

Proof. Place the tips of the gems in two opposing corners of the square, like in Figure 5.10. This placement constitutes a valid packing because (1) the gems fit into the square individually and (2) the gems never overlap. This can be proven algebraically in exactly the same fashion as in Lemma 9, where we proved a similar statement about packing into gems. □

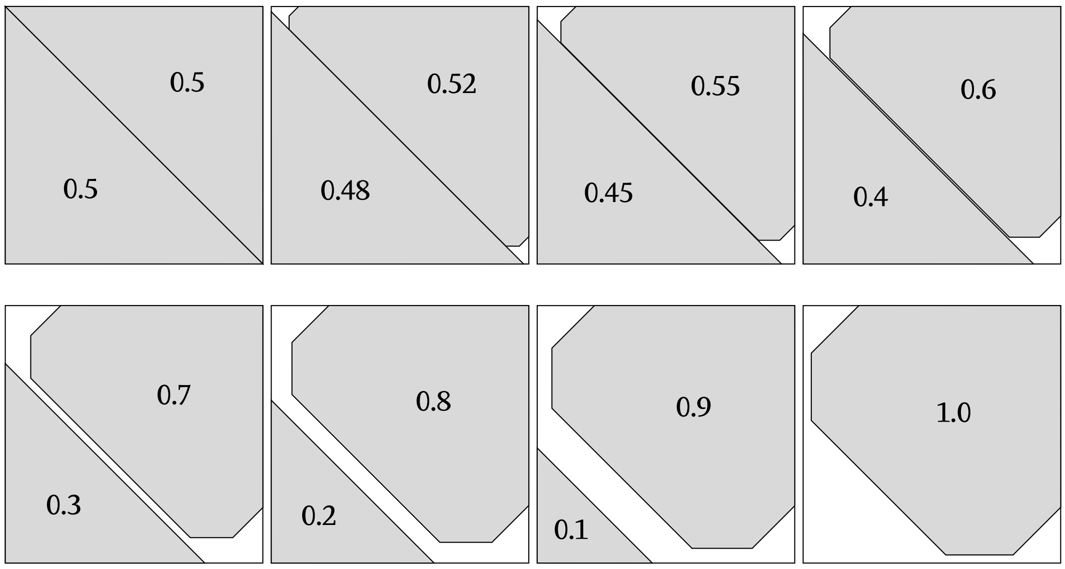


Figure 5.10: Gem-in-square packings for different values of a_1 and a_2

Theorem 12. *Given a square with a twincircle area of a, all ruby instances with a combined incircle area of up to a can be packed into the square, and this area bound is tight. See Figure 5.11 for some example packings. Expressed algebraically, the critical density for the rubies' areas is*

$$8\sqrt{2(\sqrt{2}-1)}+6\sqrt{2}-15 \approx 76.67\%.$$

Proof. By Lemma 13 and the Split Packing Theorem, the square is a $\mathbb{R}(a)$-shape.

Two rubies with a combined incircle area of more than a will have incircles which are larger than the square's twincircles. As shown in Lemma 11, those cannot be packed, which means that the rubies cannot be packed, as well.

The critical density can be calculated using the area of the rubies' incircles (see Lemma 12). □

At this point, we can make a similar observation like for isosceles right triangles: Given a square with a twincircle area of a, and some arbitrary objects, if one can enclose each object in a ruby, and the combined incircle area of these rubies is not larger than a, then the objects can be packed into the square. This is a very general result which can be used to directly solve some other interesting packing problems:

Corollary 2. *Given a square with a twincircle area of a, the following object instances can be packed into the square:*

- *all circle instances with a combined area of up to a,*
- *all octagon instances with a combined area of up to $\frac{8\sqrt{2}-8}{\pi}a \approx 1.0548a$, and*
- *all square instances with a combined area of up to $\frac{3+2\sqrt{2}}{2\pi}a \approx 0.9276a$, which is also exactly half of the square's area.*

All of these area bounds are tight. See Figure 5.12 for some example packings.

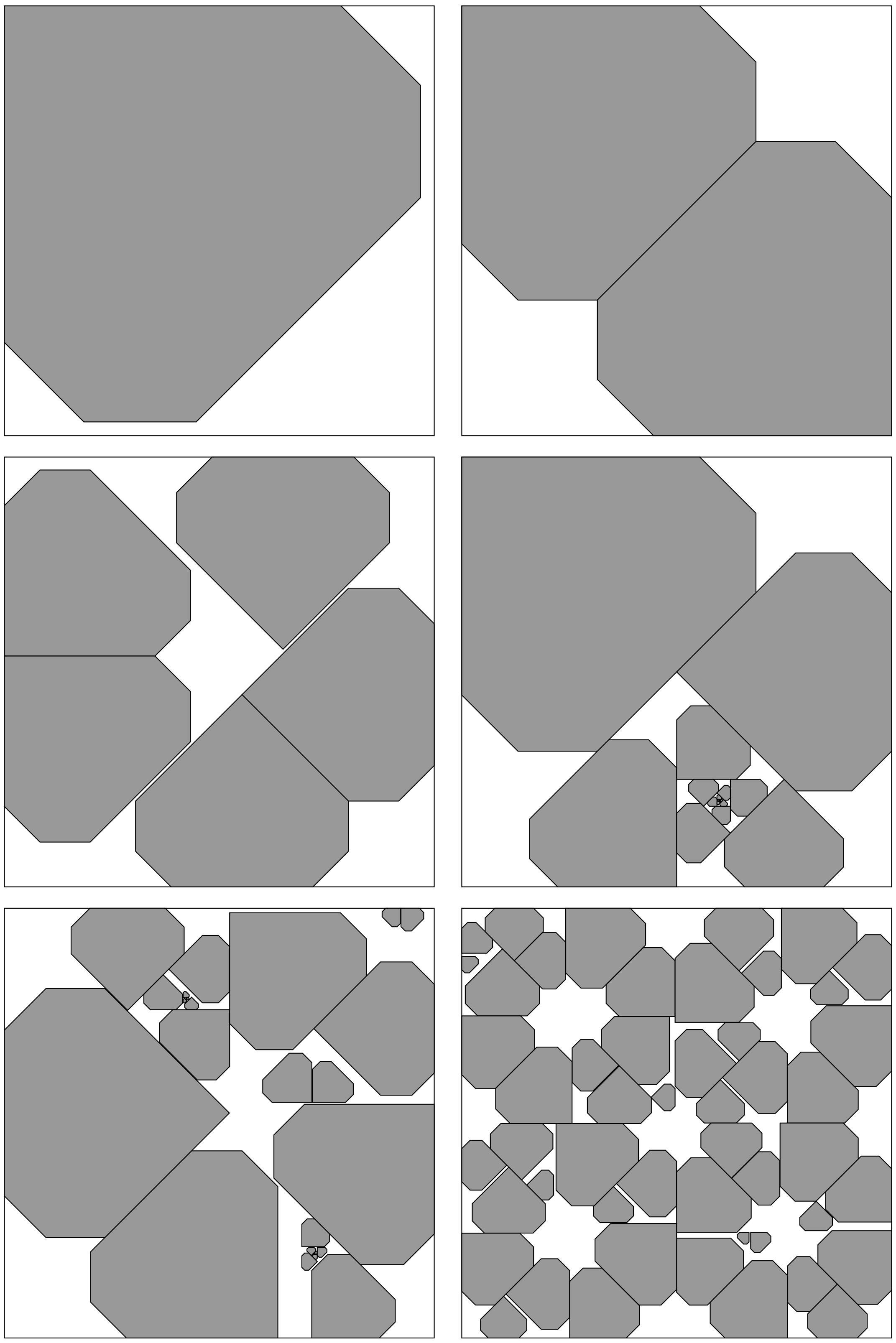


Figure 5.11: Example packings of various ruby instances in a square produced by Split Packing.

Proof. As demonstrated above, the three shapes are subsets of rubies in these specific area ratios. To pack them, enclose all of them in the smallest possible ruby. The combined incircle area of these rubies will now be at most a. We then know from Theorem 12 that we can always pack these rubies.

The tightness of these bounds for circles was shown in Lemma 12. For the same reason, two octagons with larger incircles cannot be packed: Their incircles would overlap, which means that the octagons themselves would overlap. For squares, the fact that the largest two squares that can be packed into a square (with rotation) are two quarter-squares was shown by Göbel [15]. □

Note that the result that you can pack all sets of squares with a density of up to $1/2$ is already known, even without rotations, as it can be done with Moon & Moser's Shelf Packing approach [34]. For circles and octagons, to our knowledge, this is a new result.

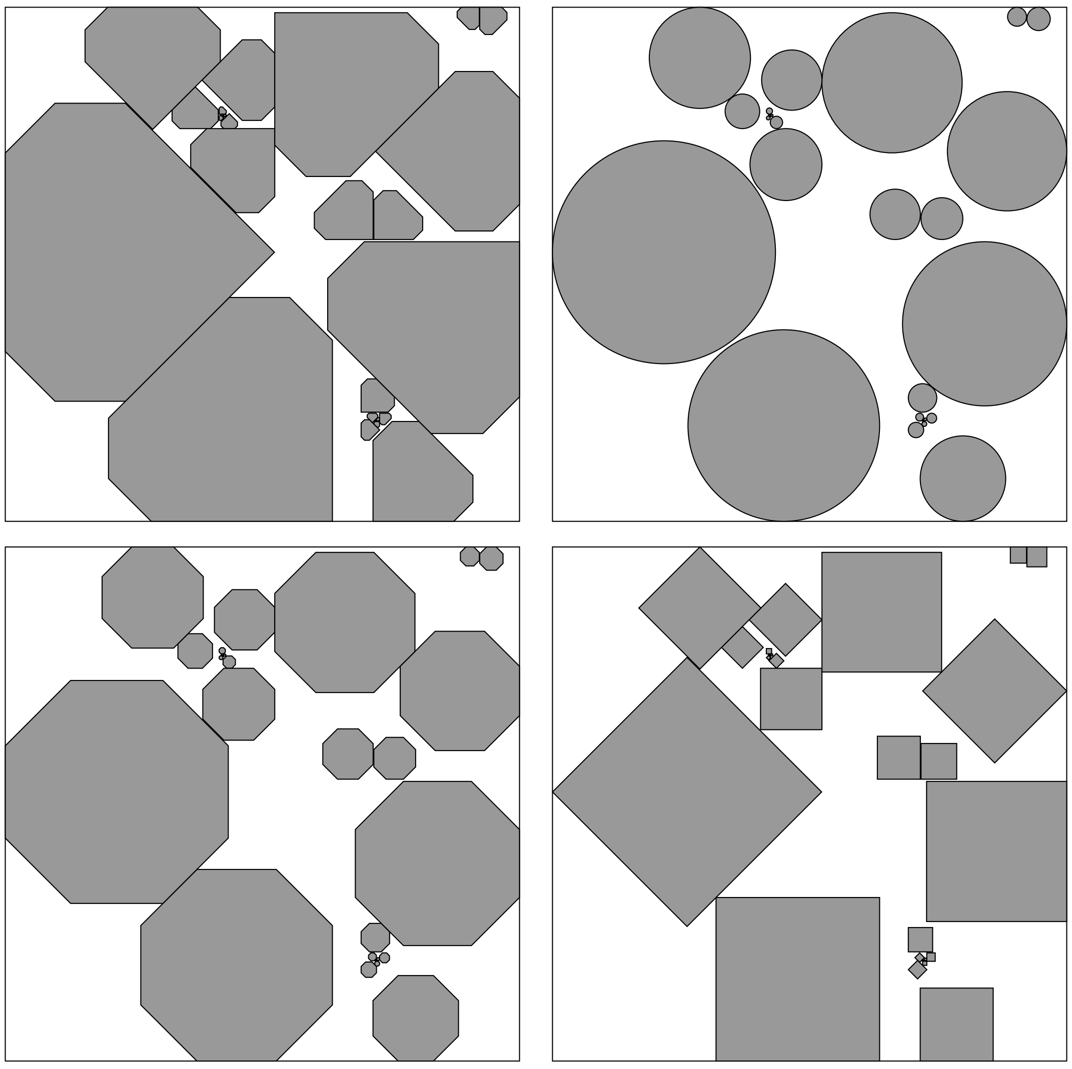


Figure 5.12: Example packings of ruby-, circle-, octagon-, and square instances in a square produced by Split Packing.

6 Future work

We see many opportunities to apply the techniques behind Split Packing in the context of other packing and covering problems.

In this thesis, we mainly considered triangular and square containers. It would be interesting to see whether the Split Packing approach could be adapted to even **more container types**, like circles, ovals, regular polygons, or generalized quadrilaterals. For some of these container types, even the worst-case instance does not seem obvious. For circular and "almost square" rectangular containers, we assume the worst cases would again be their twincircles, see Figure 6.1, but it is unclear how to deal with the resulting shapes when cutting along the circles' tangent: Compared to triangular and square containers, these shapes cannot be split into self-similar pieces. It is also possible that the depicted instances are not the actual worst cases.

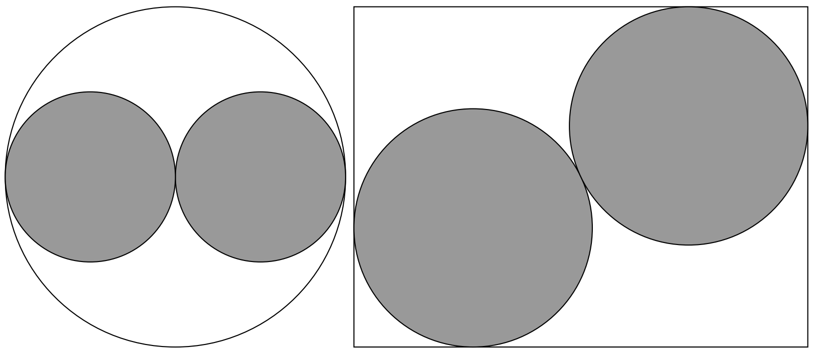


Figure 6.1: Assumed worst-case instances for a circle and a near-square rectangle.

Also, the problem of finding the critical density for packing into **acute triangles** is still open. See Section 4.4 for a discussion on why the Split Packing approach does not directly work for acute ones. A strategy for packing acute triangles with critical density, combined with the results of this thesis, would give an elegant, general result for all triangles.

Instead of packing circular, square, and octagonal objects, **more object types** could be considered, like ovals, rectangles, or even more general convex objects. For these modified problems, again, it does not seem obvious what the worst-case packings would look like.

For long rectangles (and long containers in general), in Section 4.5 we discussed that the maximally packable area is limited by the relatively small height, making the problem rather trivial. It would be a more interesting problem to require a **maximum object size**: What is the largest a so that all circle instances with a combined area of at most a, each of which have a diameter *not larger than a rectangle's smaller side*, can be packed into that rectangle?

The Split Packing method as described in this thesis is a strictly offline approach, as the greedy splitting requires sorting the complete input instance, and then proceeds in a top-down fashion. It would be interesting to study to which extent the obtained results could be replicated in an **online situation**. The best algorithm that packs squares into a square in an online fashion is currently by Brubach [6] and gives a density guarantee of $\frac{2}{5}$. As mentioned in Section 2.1, we can directly use this algorithm to pack circles into a square in an online situation with a density of $\frac{\pi}{10} \approx 0.3142$. It would be particularly interesting to see whether some form of online Split Packing would give better results.

Our original motivation stemmed from origami design. When only packing circles, the resulting origami structures resemble arbitrary stars. When one wants to design general tree-shaped structures, it is necessary to introduce separating pathways between the circles, a technique called **circle/river packing**, pioneered by Lang [26]. A packing scheme like Split Packing seems promising because it often introduces gaps inbetween two subgroups anyway. At this point, we can establish a constant-factor approximation for perfectly symmetric binary trees (see Figure 6.2), but we do not know how to approximate the paper size needed for crease patterns of general trees.

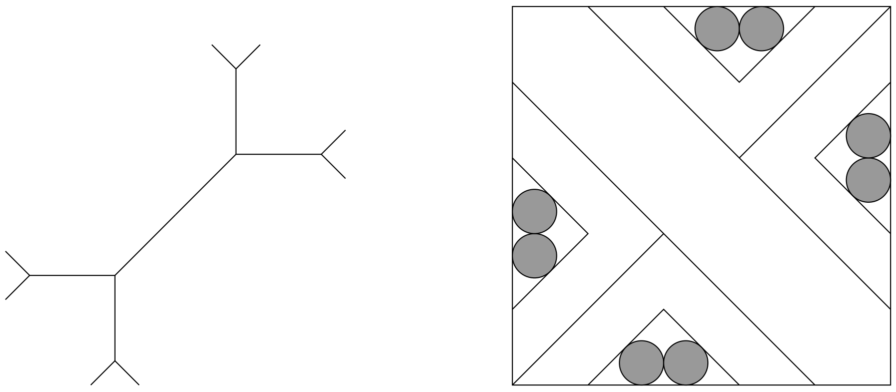


Figure 6.2: A folding of the tree on the left can be realized by a crease pattern based on the circle/river packing on the right.

It seems like a natural extension to apply Split Packing to **three-dimensional packing** problems. For example, one could try to pack spheres into a cube using a Split Packing approach. Unfortunately, this does not directly seem to work out: Assuming the worst case are two equally sized spheres packed into opposite corners of the cube, one would like to be able to cut the cube along the spheres' tangential plane. This results in two shapes as depicted on the right in Figure 6.3, but it is not possible to fit two quarter-spheres into each of these polyhedra. Still, any extensions regarding three dimensional problems would be notable.

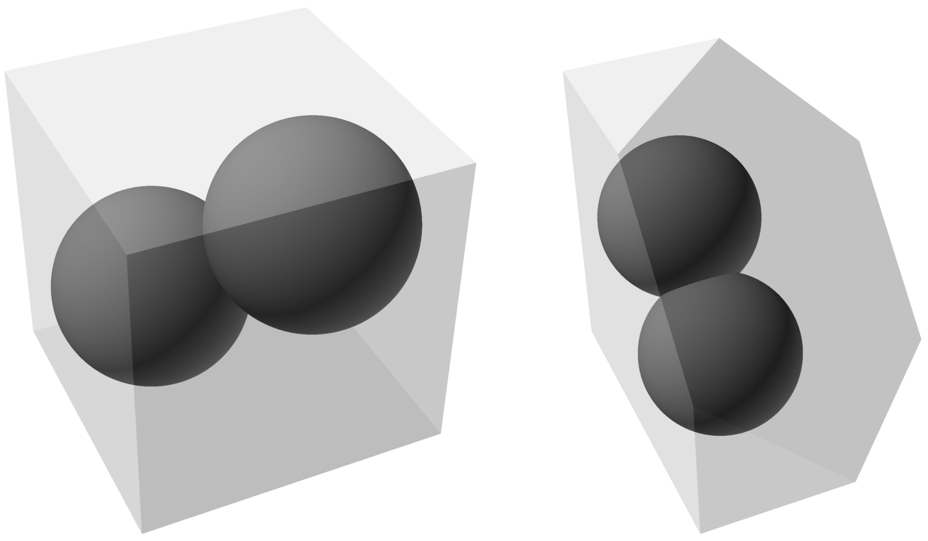


Figure 6.3: Left: Assumed worst case for packing spheres into a cube. Right: Two quarter-spheres do not fit in a half.

Instead of packing circles into containers, one could ask a question which is in some sense the opposite problem: What is the smallest area so that we can always **cover the container** with circles of that combined area? For example, if we want to cover an isosceles right triangle, and restrict ourselves to at most two circles in our input instance, the area of a circle whose diameter equals the triangle's hypotenuse is sufficient, see Figure 6.4. It would now be necessary to show that the area of the left circle is always enough to cover the quadrilateral on the left, and it does not seem trivial to find an argument for that.

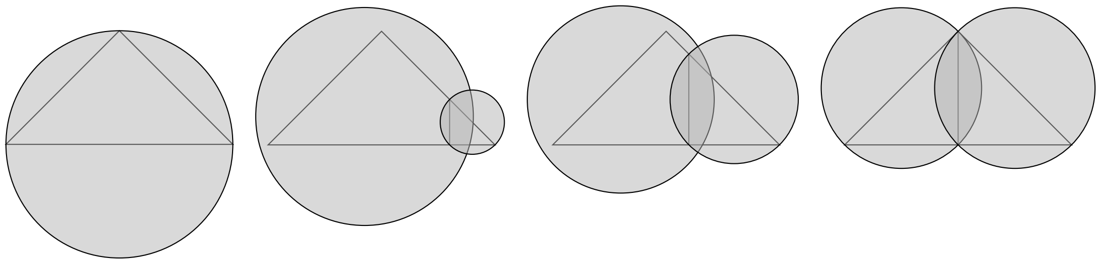


Figure 6.4: An isosceles right triangle can always be covered by two circles with a combined area of its excircle.

Bibliography

[1] B. Addis, M. Locatelli, and F. Schoen. “Disk Packing in a Square: A New Global Optimization Approach”. In: *INFORMS Journal on Computing* 20.4 (Nov. 2008), pp. 516–524. ISSN: 1526-5528. DOI: 10.1287/ijoc.1080.0263.

[2] Ernesto G. Birgin and Jan M. Gentil. “New and improved results for packing identical unitary radius circles within triangles, rectangles and strips”. In: *Computers & Operations Research* 37.7 (July 2010), pp. 1318–1327. ISSN: 0305-0548. DOI: 10.1016/j.cor.2009.09.017.

[3] Ernesto G. Birgin and F. N. C. Sobral. “Minimizing the object dimensions in circle and sphere packing problems”. In: *Computers & Operations Research* 35.7 (2008), pp. 2357–2375.

[4] David W. Boll, Jerry Donovan, Ronald L. Graham, and Boris D. Lubachevsky. “Improving dense packings of equal disks in a square”. In: *The Electronic Journal of Combinatorics* 7.1 (2000), R46.

[5] Paul Bourke. *Random space filling tiling of the plane.* July 2011. URL: http://paulbourke.net/texture_colour/randomtile/.

[6] Brian Brubach. “Improved Bound for Online Square-into-Square Packing”. In: *Approximation and Online Algorithms.* Springer, 2014, pp. 47–58.

[7] Ignacio Castillo, Frank J. Kampas, and János D. Pintér. “Solving circle packing problems by global optimization: numerical results and industrial applications”. In: *European Journal of Operational Research* 191.3 (2008), pp. 786–802.

[8] Hai-Chau Chang and Lih-Chung Wang. “A Simple Proof of Thue’s Theorem on Circle Packing”. In: *arXiv preprint arXiv:1009.4322v1 [math.MG]* (Sept. 22, 2010). arXiv: 1009.4322v1 [math.MG].

[9] Erik D. Demaine, Sándor P. Fekete, and Robert J. Lang. “Circle Packing for Origami Design is Hard”. In: *5th International Conference on Origami in Science, Mathematics and Education.* AK Peters/CRC Press. 2011, pp. 609–626.

[10] László Fejes Tóth. “Über einen geometrischen Satz”. In: *Mathematische Zeitschrift* 46.1 (Dec. 1940), pp. 83–85. DOI: 10.1007/BF01181430.

[11] Juan J. Flores, Jose Martínez, and Felix Calderón. “Evolutionary computation solutions to the circle packing problem”. In: *Soft Computing* (Feb. 2015). ISSN: 1433-7479. DOI: 10.1007/s00500-015-1603-y.

[12] Robert J. Fowler, Michael S. Paterson, and Steven L. Tanimoto. “Optimal packing and covering in the plane are NP-complete”. In: *Information Processing Letters* 12.3 (June 1981), pp. 133–137. ISSN: 0020-0190. DOI: 10.1016/0020-0190(81)90111-3.

[13] Hamish J. Fraser and John A. George. “Integrated container loading software for pulp and paper industry”. In: *European Journal of Operational Research* 77.3 (1994), pp. 466–474.

[14] John A. George, Jennifer M. George, and Bruce W. Lamar. "Packing different-sized circles into a rectangular container". In: *European Journal of Operational Research* 84.3 (Aug. 1995), pp. 693–712. ISSN: 0377-2217. DOI: 10.1016/0377-2217(95)00032-l.

[15] F. Göbel. "Geometrical packing and covering problems". In: *Packing and Covering in Combinatorics, A. Schrijver (ed.), Math Centrum Tracts* 106 (1979), pp. 179–199.

[16] R. L. Graham, B. D. Lubachevsky, K. J. Nurmela, and P. R. J. Östergård. "Dense packings of congruent circles in a circle". In: *Discrete Mathematics* 181.1-3 (Feb. 1998), pp. 139–154. ISSN: 0012-365X. DOI: 10.1016/s0012-365x(97)00050-2.

[17] Ronald L. Graham and Boris D. Lubachevsky. "Repeated patterns of dense packings of equal disks in a square". In: *The Electronic Journal of Combinatorics* 3.1 (1996), R16.

[18] Claas de Groot, Ronald Peikert, and D. Würtz. *The optimal packing of ten equal circles in a square.* 1990.

[19] A. Grosso, A. R. M. J. U. Jamali, M. Locatelli, and F. Schoen. "Solving the problem of packing equal and unequal circles in a circular container". In: *Journal of Global Optimization* 47.1 (2010), pp. 63–81.

[20] Kun He, Menglong Huang, and Chenkai Yang. "An action-space-based global optimization algorithm for packing circles into a square container". In: *Computers & Operations Research* 58 (June 2015), pp. 67–74. ISSN: 0305-0548. DOI: 10.1016/j.cor.2014.12.010.

[21] Mhand Hifi and Rym M'hallah. "A literature review on circle and sphere packing problems: models and methodologies". In: *Advances in Operations Research* (2009).

[22] Pedro Hokama, Flávio K. Miyazawa, and Rafael C. S. Schouery. "A bounded space algorithm for online circle packing". In: *Information Processing Letters* 116.5 (May 2016), pp. 337–342. ISSN: 0020-0190. DOI: 10.1016/j.ipl.2015.12.007.

[23] Stefan Hougardy. "On packing squares into a rectangle". In: *Computational Geometry* 44.8 (2011), pp. 456–463. ISSN: 0925-7721. DOI: http://dx.doi.org/10.1016/j.comgeo.2011.05.001.

[24] Wen Qi Huang, Yu Li, Chu Min Li, and Ru Chu Xu. "New heuristics for packing unequal circles into a circular container". In: *Computers & Operations Research* 33.8 (Aug. 2006), pp. 2125–2142. ISSN: 0305-0548. DOI: 10.1016/j.cor.2005.01.003.

[25] Daniel J. Kleitman and Michael M. Krieger. "An optimal bound for two dimensional bin packing". In: *Foundations of Computer Science, 1975., 16th Annual Symposium on* (1975), pp. 163–168.

[26] Robert J. Lang. "A computational algorithm for origami design". In: *Proceedings of the twelfth annual symposium on Computational geometry - SCG '96* (1996). DOI: 10.1145/237218.237249.

[27] Joseph Y. T. Leung, Tommy W. Tam, Chin S. Wong, Gilbert H. Young, and Francis Y. L. Chin. "Packing squares into a square". In: *Journal of Parallel and Distributed Computing* 10.3 (1990), pp. 271–275.

[28] Igor Litvinchev, Luis Infante, and Edith Lucero Ozuna Espinosa. "Approximate Circle Packing in a Rectangular Container: Integer Programming Formulations and Valid Inequalities". In: *Computational Logistics* (2014), pp. 47–60. ISSN: 1611-3349. DOI: 10.1007/978-3-319-11421-7_4.

[29] Marco Locatelli and Ulrich Raber. "Packing equal circles in a square: a deterministic global optimization approach". In: *Discrete Applied Mathematics* 122.1 (2002), pp. 139–166.

[30] C. O. López and J. E. Beasley. "Packing unequal circles using formulation space search". In: *Computers & Operations Research* 40.5 (May 2013), pp. 1276–1288. ISSN: 0305-0548. DOI: 10.1016/j.cor.2012.11.022.

[31] Boris D. Lubachevsky and Ronald L. Graham. "Curved hexagonal packings of equal disks in a circle". In: *Discrete & Computational Geometry* 18.2 (1997), pp. 179–194.

[32] Mihály Csaba Markót and Tibor Csendes. "A new verified optimization technique for the" packing circles in a unit square" problems". In: *SIAM Journal on Optimization* 16.1 (2005), pp. 193–219.

[33] Flávio K. Miyazawa, Lehilton L. C. Pedrosa, Rafael C. S. Schouery, Maxim Sviridenko, and Yoshiko Wakabayashi. "Polynomial-Time Approximation Schemes for Circle Packing Problems". In: *Lecture Notes in Computer Science* (2014), pp. 713–724. ISSN: 1611-3349. DOI: 10.1007/978-3-662-44777-2_59.

[34] John W. Moon and Leo Moser. "Some packing and covering theorems". In: *Colloquium Mathematicae*. Vol. 17. 1. Institute of Mathematics Polish Academy of Sciences. 1967, pp. 103–110.

[35] J. K. Nurmela and J. P. R. Östergård. "More Optimal Packings of Equal Circles in a Square". In: *Discrete & Computational Geometry* 22.3 (1998), pp. 439–457. ISSN: 1432-0444. DOI: 10.1007/PL00009472.

[36] Ronald Peikert, Diethelm Würtz, Michael Monagan, and Claas de Groot. *Packing circles in a square: a review and new results*. Springer, 1992.

[37] J. Schaer. "The densest packing of nine circles in a square". In: *Canad. Math. Bull* 8 (1965), pp. 273–277.

[38] J. Schaer and A. Meir. "On a geometric extremum problem". In: *Canad. Math. Bull* 8.1 (1965).

[39] Eckard Specht. *Packomania*. 2015. URL: http://www.packomania.com/.

[40] Kokichi Sugihara, Masayoshi Sawai, Hiroaki Sano, Deok-Soo Kim, and Donguk Kim. "Disk packing for the estimation of the size of a wire bundle". In: *Japan Journal of Industrial and Applied Mathematics* 21.3 (2004), pp. 259–278.

[41] Péter Gábor Szabó, Mihaly Csaba Markót, Tibor Csendes, Eckard Specht, Leocadio G. Casado, and Inmaculada García. *New Approaches to Circle Packing in a Square*. Springer US, 2007. DOI: 10.1007/978-0-387-45676-8.

[42] Axel Thue. "Om nogle geometrisk-taltheoretiske theoremer". In: *Forandlingerneved de Skandinaviske Naturforskeres* 14 (1892), pp. 352–353.

[43] Huaiqing Wang, Wenqi Huang, Quan Zhang, and Dongming Xu. "An improved algorithm for the packing of unequal circles within a larger containing circle". In: *European Journal of Operational Research* 141.2 (Sept. 2002), pp. 440–453. ISSN: 0377-2217. DOI: 10.1016/s0377-2217(01)00241-7.

[44] D. Würtz, M. Monagan, and R. Peikert. "The history of packing circles in a square". In: *Maple Technical Newsletter* (1994), pp. 35–42.

[45] De-fu Zhang and An-sheng Deng. "An effective hybrid algorithm for the problem of packing circles into a larger containing circle". In: *Computers & Operations Research* 32.8 (Aug. 2005), pp. 1941–1951. ISSN: 0305-0548. DOI: 10.1016/j.cor.2003.12.006.